Number Sense

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Preface

The first nine sections of this resource are from a graduate level literature review written by C. David Pilmer. The author gives the Nova Scotia Department of Labour and Advanced Education permission to reproduce this material for professional development purposes in the Adult Learning Branch. The activity sheets found in the appendices of this resource are the property of the Department of Labour and Workforce Development. Instructors/teachers are permitted to reproduce these activity sheets for use in their classrooms.

This resource is to be used by Level II, III and IV Math instructors within the Nova Scotia School for Adult Learning (NSSAL). It was designed to give instructors a general understanding of number sense and how number sense activities can be incorporated into their teaching and assessment practices. It also includes numerous black line masters (BLM) which instructors can reproduce for classroom purposes. Instructors will have to decide which BLMs are appropriate for their students. Some of these sheets have a part 1 and part 2. In many cases, learners are not expected to complete both parts within the same course. Typically the part 1 sheets work with whole numbers, and are more appropriate for Level I and II learners. The part 2 sheets often deal with fractions and integers, and this are more appropriate for Level III and IV learners. The following chart states the general purpose of each sheet and the appropriate level.

Title of Sheet	**Purpose**	**Level**
Expressing a Number Different Ways	Flexibility with Number	I & II
Sum and Product Squares (Part 1)	Flexibility with Number	I & II
Sum and Product Squares (Part 2)	Flexibility with Number	III & IV
Picking a Route	Flexibility with Number	I & II
Blazing a Trail (Part 1)	Flexibility with Number	I & II
Blazing a Trail (Part 1)	Flexibility with Number	III & IV
Bull's Eye	Flexibility with Number	II
The Fantastic Four Card Game	Flexibility with Number	II & III
Sequences (Part 1)	Flexibility with Number	II & III
Sequences (Part 2)	Flexibility with Number	III & IV
What Portion is Shaded?	Proportional Reasoning	II & III
Fraction, Decimal and Percent Cards	Proportional Reasoning	II & III

Title of Sheet	Purpose	Level
Proportional Reasoning Squares	Proportional Reasoning	II & III
Approximately How Full?	Proportional Reasoning	II & III
Finding Numbers Between Other Numbers	Proportional Reasoning	III & IV
The Number Line (Part 1)	Number Magnitude	II
The Number Line (Part 2)	Number Magnitude	III & IV
Just the Answer	Mental Math	I & II
Mental Computations	Mental Math	IV
Estimating by Comparing Objects	Estimation	I & II
Classify	Estimation	III
Reasonable Estimates?	Estimation	IV
Do It In Your Head (Part 1)	Mental Math and Estimation	II & III
Do It In Your Head (Part 2)	Mental Math and Estimation	III & IV

The resource is not meant to be the definitive work on number sense, merely an introduction. Many of you will find that the resources Number Sense; Grades 4 -6 (McIntosh, Reys, Reys & Hope, 1997) and Number Sense; Grades 6 - 8 (McIntosh, Reys & Reys, 1997) provide a greater number of activities that are more appropriate for your learners. Another resource that is highly recommended is Teaching Student-Centered Mathematics: Grades 5-8 (Van de Walle & Lovin, 2006). These three resources should be used in conjunction with this Department of Education resource.

As you read through this document, you will revisit three common themes regarding number sense.

(1) Fostering number sense can ultimately improve a learner's acquisition of future mathematical concepts.
(2) Number sense activities should be incorporated into daily teaching practices. It should not be viewed as a separate unit, taught in isolation of other units or concepts.
(3) Acquiring number sense is a gradual process, where learners often use different but valid strategies, and demonstrate different levels of sophistication.

Many of the research articles cited in this resource recommend the use of the constructivist or discovery approach in the development of number sense. Much of this research was conducted during the 1990's and little time had been spent examining the effects of this type of learning on low-achieving math students or students with a math disability. More recent research recommends explicit teacher-directed instruction, peer-assisted learning and scaffolded investigations for low-achieving students. Similar research contends that even scaffolded investigations can be problematic for students with a math disability. This is important to keep this in mind as you read through the document.

What is Number Sense?

Number sense differs from student to student. Consider the following example. All three students have been asked the same estimation problem and all have answered 23; a very valid estimation. Only after further discussions do we realize that they have taken different approaches.

Estimate: $\frac{1}{3} \times 9.5 + 10.4 \div 0.51$

Student 1 $0.3 \times 10 + 10 \div 0.5$ $3 + 20$ 23	• I knew that $\frac{1}{3}$ is approximately equal to 0.3. • Using the rules for rounding, I changed 9.5 to 10, 10.4 to 10, and 0.51 to 0.5. • three-tenths of 10 is 3. • 100 divided by 5 is twenty, so 10 divided by 0.5 must also be twenty. • 3 plus 20 is 23
Student 2 $\frac{1}{3} \times 9 + 10 \div \frac{1}{2}$ $3 + 20$ 23	• Change 9.5 to 9 because it is easier to multiply 9 by $\frac{1}{3}$, versus 10. • 0.51 is approximately equal to $\frac{1}{2}$. • If there are two halves in one, there must be twenty halves in 10. • One-third of 9 is 3. • Add 3 and 20, you get 23
Student 3 $0.3 \times 10 + 10 \div \frac{1}{2}$ $3 + 10 \times \frac{2}{1}$ $3 + 20$ 23	• I knew that $\frac{1}{3}$ is approximately equal to 0.3. • Using the rules for rounding, I changed 9.5 to 10, and 10.4 to 10. • 0.51 is approximately equal to $\frac{1}{2}$. • Dividing by a fraction is the same as multiplying by its reciprocal. • The sum of 3 and 20 is 23.

All three students show evidence of number sense but of varying degrees.

So what is number sense? Over the last twenty years, extensive work has been done in the area of number sense. Researchers such as Paul Trafton, Zvia Markovits, Judith Sowder, Robert Reys, Barbara Reys, Bonnie Schappelle, John Hope, Micheal Forrester, Alister McIntosh Thomas Carpenter, James Greeno, and Lauren Resnick consistently write about or are referenced in articles concerned with number sense. Many of these individuals have worked together on numerous projects, articles and teacher resources so it is not surprising that their definitions of number sense are remarkably similar. Their definitions only vary slightly but each serves to compliment the definition proposed by other researchers.

Based on the work of these researchers, three critical common components for defining number sense can be identified.

(1) Number sense is a sound understanding of numbers and operations (MacIntosh, Reys, and Reys, 1997, National Council of Teachers of Mathematics, 1989, Thompson & Rathmall as cited in Schappelle & Sowder, 1989). This means that students can move between number representations (e.g. decimals, fractions, percents, scientific notation), recognize number magnitude, and relate number, symbols, and operations. In the following example the student has displayed a sound understanding of operation when correctly completing the following mental computation.

Example:

Evaluate: $24+24+24$

Student's Strategy and Explanation:

$24+24+24$	
3×24	• Change from addition to multiplication.
$3\times(25-1)$	• It's easier to multiply by 25 so change 24 into 25-1.
$75-3$	• I multiplied through.
72	• I did the subtraction.

This student clearly understood the relationship between addition and multiplication and although he/she was not able to use the standard terminology, he/she appeared to be comfortable using the distributive property.

(2) Number sense is the ability to operate flexibly with number (Carpenter, 1989, Greeno, J. G., 1991, Markovits & Sowder, 1994, MacIntosh, Reys & Reys 1997, Plunkett as cited in MacIntosh, Reys & Reys, 1979, Reys and Yang, 1998, Schappelle & Sowder, 1989). This means that students do not feel compelled to use algorithms, rather they are occupied with finding the most useful, efficient, and sometimes even unconventional approach to handling the problem.

(3) Number sense is characterized by its intuitive nature (Greeno as cited by Markovits and Sowder, 1994, Howden, 1989, National Council of Teachers of Mathematics, 1989). In the career of any classroom mathematics teacher, we have recognized that some students display an almost effortless ability to gain insight into a problem and identify which strategy or strategies could be used.

If one considers the three criteria above as the bare bones of number sense, the following traits, though not mentioned by all researchers, contribute to a richer understanding of the term.

(1) Number sense develops gradually (Howden, 1989). If students possess the inclination and are provided opportunity to explore numbers both in their formal and informal education, number sense can likely be fostered. Number sense is possessed at different levels for different individuals however these levels have the potential for constantly expanding, reflecting the new experiences they encounter and the new insights they obtain (Reys, 1989). Number sense also "develops and matures with experience and knowledge" (Reys as cited in Reys & Yang, 1998). This does not mean however, that with time alone all students will develop number sense (McIntosh, Reys, Reys & Hope, 1997).

(2) Number sense is highly personal. Not only is it related to what ideas about number have been established but also to how those ideas were established (McIntosh, Reys, Reys & Hope, 1997). The statement implies that the nature of classroom instruction and activities is critical in the development of number sense. Number sense develops "as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways not limited by traditional algorithms... (and that it) ...builds on students' natural insights and convinces them that mathematics makes sense." (Howden, 1989) This statement suggests that traditional teaching practices are not always conducive to developing number sense. Number sense is a by-product of teaching for understanding. If students experience number rather than simply work with number, they gain insight and meaning, and make connections. These opportunities draw from a child's informal knowledge and in turn fosters number sense (Carpenter citied in Schapelle & Sowder, 1989).

(3) Number sense is dependent on a complex interaction among an individual's knowledge, an individual's skills, the nature of the problem and the expected performance on a particular problem. Number sense also involves judgement and interpretation which provides greater elaboration to notion of flexibility with number (Resnick, 1989).

(4) Number sense can yield multiple methods and/or solutions. (Markovits, 1989, Resnick, 1989). This statement can be expanded upon by saying that students relate numbers in ways not limited to traditional algorithms (Howden, 1989). If one considers the three student responses to the estimation problem at the beginning of this paper, one can see that different methods were employed even though the same solution was obtained. The thinking strategies employed by individual students vary in efficiency and elegance depending on the sophistication of the student's understanding (MacIntosh, Reys and Reys, 1997).

The Major Components of Number Sense

If one is attempting to judge the nature of number sense possessed by an individual, one must examine the flexibility with number displayed by the individual. This flexibility can be observed when the students are completing items from the four major components of number sense; judging number magnitude, mentally computing, estimating, judging reasonableness of results (Markovits & Sowder, 1994, McIntosh, Reys, Reys & Hope, 1997). Many of the teacher resources on number sense available today allow students to explore these four areas in an attempt to allow students to make sense of number and in turn foster number sense.

Understanding number magnitude means that individuals should be able to compare numbers such that they can order the numbers, recognize which of two numbers is closer to a third, and to identify numbers between two given numbers (Markovits & Sowder, 1994). It is desired that they be able to accomplish this while comparing different representations of numbers (McIntosh, Reys, Reys & Hope, 1997). This means students should feel comfortable working with whole numbers, decimals, fractions, percents, and exponents either together or in isolation.

Examples:

Order $0.4, \frac{1}{5}, \frac{8}{7}, 0.09$

Is $\frac{5}{8}$ or $\frac{7}{12}$ closer to 0.5?

Determine a fraction between $\frac{4}{9}$ and $\frac{5}{9}$.

Mental computation is the process of calculating the exact numerical answer without the aid of any external calculating or recording device. Student perception regarding what it means to compute mentally differ greatly. Some students believe that you are merely using the prescribed algorithms mentally (MacIntosh, Reys & Reys, 1997). Consider the following example and the two strategies employed by different students.

Student 1: standard algorithm done mentally

$$\begin{array}{r} {}^{1} \\ 6.4 \\ +\ \ 1.9 \\ \hline 8.3 \end{array}$$

Student 2: thinking strategy

$6.4 + 1.9$

$6.4 + 2 - 0.1$

$8.4 - 0.1$

8.3

Mental computation can include algorithms but alternate strategies should also be encouraged/supported such as the one proposed by Student 2 in the previous example.

Estimation can be broken into three distinct categories; numerosity, measurement, and computational estimation (Hanson & Hogan, 2000). Numerosity refers to one's ability to estimate the number of objects present. A student could be asked to estimate how many pencils have been scattered on the floor. Measurement refers to one's ability to estimate the weight, length or volume of an object or the time required to complete a task. Computational estimation refers to one's ability to estimate answers to numerical computations. This paper will limit itself to computational estimation.

Examples:

Estimate 16% of 48.

Estimate $357 + 51.3 + 0.67 - 11.3$

Estimate $305 \div 0.312$

Estimate $\frac{4}{7}+\frac{5}{11}+\frac{17}{16}$.

Judging the reasonableness of a result means that students should examine the answer they have obtained with or without technology and determine whether the answer is appropriate given the question and the context.

Examples:

When you multiply 13.26 and 3.5, the answer is 4641, but the decimal point is missing. Place the decimal point in the appropriate position.

A school bus can carry 40 students. If 98 students and 5 teachers are going on a fieldtrip, how many buses are required?

Why should teachers foster number sense? Research of adult usage of mathematics demonstrates that approximately 80% of mathematical computations in daily life require the mental manipulation of numerical quantities rather than the usage of traditional algorithms (Edwards cited in Reys & Reys, 1995). Number sense is a necessity in life. In a mathematically literate society individuals can think flexibly with numbers, whether they are mentally calculating the best value at the grocery store, estimating the return of money market funds, or checking the reasonableness of a calculator result (Reys & Reys, 1995). What is encouraging is that the research demonstrates that number sense can be developed and fostered in the classroom.

This paper will examine how number sense can be fostered in a classroom and specifically examine the four major components of number sense: number magnitude, mental computation, computational estimation, and reasonableness of answers. In addition to this, the paper will supply a variety of questions which may be used by the teacher, numerous strategies that may be employed by students and address issues regarding assessment of number sense.

Fostering Number Sense

Conceptions of Mathematics

The sense-making of mathematics is critical to the development of number sense. Students who possess strong number sense recognize and flexibly apply relationships between numbers and between operations. While they may use algorithms efficiently and with a level of expertise, they have made sense of mathematics and, in turn, have found non-standard approaches to solve questions in a manner, which are efficient for themselves. Unfortunately many students don't believe that math is supposed to make sense or be logical. These students believe that mathematics is a series of practiced algorithms and that the discipline is devoid of questioning, of creativity, and of sense-making (Phillip, Flores, Sowder, & Schappelle as cited in Sowder, 1995, Reys, 1989, Silver, 1990). This isn't surprising when one considers that approximately 85% to 95% of instructional time is devoted to mastering the use of algorithms. However, researchers suggest that this should be reduced significantly to 10% so that more time can be spent doing mental computation and estimation (Shumway as cited in McIntosh, Reys & Reys, 1997).

Learned algorithms although valued by students do not insure that students are making sense of mathematics and it does not tap into the creative elements of this discipline. Students tend to ignore their informal insights in favor of the algorithms. Unless situations are provided where students explore and experience authentic sense-making aspects of mathematics and where it is valued, it is unlikely that they will alter their belief that math is devoid of questioning, creativity, and sense-making (Silver, 1990). If students are exposed to a variety of problems designed to assist in the development of number sense, they will recognize the dynamic nature of mathematics and develop valued thinking skills.

For many teachers, their conceptions of mathematics are deeply rooted in their own formal education. Many teachers are products of very traditional practices and curriculums where instruction was designed to lead students through explicit and very systematic lessons that would ultimately lead to the mastery of content. Although this expertise is not without value, problems often arise when these individuals are exposed to novel or unusual problems where practiced procedures are not sufficient or obvious (Hatano as cited in Markovits & Sowder, 1994). The teachers lacked the necessary conceptual knowledge required to address the problems because they had failed to explore the concepts and construct their own meaning and connections. A teacher's conceptions of mathematics teaching and learning are formed during their own schooling years and are influenced by their own mathematical experiences during those times (Ball as cited in Thompson, 1992). Interestingly enough, research has not demonstrated that there is a consistent relationship between a teacher's professed beliefs and the instructional practices they select (Brown as cited in Thompson, 1992). While some studies have reported a high degree of agreement, others have not. Nonetheless the question arises, if long-held, deeply rooted traditional teacher beliefs of mathematics may effect instructional practices and in turn hinder the development of number sense, how can these teacher beliefs be modified? One approach is to use resources that force teachers to reexamine their own beliefs. This is essential for gradually moving from one stage to the next. These resources should be inundated with novel problem-solving situations that cause confusion, doubt and controversy. For teachers inclined to view mathematics as a static rather than dynamic discipline, challenging their beliefs and strategies is imperative (Thompson, 1992). As addressed later in this paper, this is the same process that can be used to alter student conceptions of mathematics.

Research has shown that a teacher's knowledge and awareness of conceptual interconnections is not influenced by the number of mathematics courses they took at the post-secondary level (Eisenburg as cited in Fennema & Franke, 1992) but rather by the continuing long term commitment to math-specific professional development. Nonetheless teachers feel that they lack the necessary background to tackle such a large and pervasive topic as number sense. Teachers who are prepared to explore the dynamic nature of mathematics will develop new insights and learn to celebrate alternate and sometimes nonstandard approaches to problems. Connections are made, relationships are better understood and in turn number sense is fostered.

Teacher conceptions of mathematics not only effect how and what a student learns but it may also effect the instructional practices and materials a teacher wishes to use. Number sense is best developed in an environment that supports exploration and sense-making, which is intrinsically tied to the conception that math is "dynamic, problem-driven and continually expanding field of human creation and invention" (Ernest as cited in Thompson, 1992). There is "no substitute for a skillful teacher who fosters curiosity and exploration at all grade levels" (Howden, 1989)

Is Number Sense Assured?

Most children and adults possess some level of number sense at least with whole numbers (Trafton 1989, Hanson & Hogan 2000). Studies show however, that number sense in other areas is not insured with time. One such study by Markovits (1989) involved 49 education students who were training to become elementary school teachers. The researcher asked the education students to complete a series of questions on number sense. Here are some of the questions and results.

Question: The height of a 10-year old boy is 5 feet. What do you think his height will be when he is 20?

Result: 13% answered 10 feet (unreasonable answer)

Question: When you multiply 15.24 and 4.5, the answer is 6858, but the decimal point is missing. Place the decimal point where you think it should be.

Result: 79% answered that the decimal point should be after the six, 6.858 rather than the correct answer of 68.58.

Question: Order from smallest to largest. $0.53, \frac{14}{13}, \frac{5}{12}, 0.993$

Result: 31% were unable to order the numbers correctly.

Question: Would the answer to $264 \div 0.79$, be greater than, equal to, or less than 264.

Result: 49% answered incorrectly by saying the quotient should be less than 264. They rationalized that division always makes things smaller.

(Markovits, 1989)

These results indicate that a significant portion of these education students had difficulties estimating, computing mentally, judging number magnitude, and judging the reasonableness of answers; all critical components of number sense.

In a similar study, it was discovered that college students were competent in providing estimates for problems with whole numbers but had difficulty estimating questions involving fractions and/or decimals. In some cases, they were unable to even initiate problems involving estimation with decimals and fractions (Hanson & Hogan 2000). Both of these studies, which involved students of average to above average intelligence, demonstrated that students who successfully complete high school, do not necessary possess a high level of number sense. One can conclude that it takes more than just time and practiced algorithms to develop number sense.

Researchers also contend that number sense is not insured by using technology in mathematical pursuits. If students do not learn to use this technology judiciously and fail to consider the reasonableness of the answer supplied then number sense is compromised. Although there are situations where this technology is helpful, there are other times where students develop an over-reliance on the calculator (Silver, 1990).

Curriculum

How could curriculum be changed or altered to influence student and teacher conceptions of mathematics and also create an environment that fosters number sense? The emphasis of classroom instruction and prescribed curriculum should be the sense-making of mathematics. Classrooms conducive to sense-making, allow all students to explore and discuss mathematics in a manner where their comments are addressed and respected. In these classrooms, learning exceeds merely acquiring skills and information (Sowder, 1995) and provides students with opportunities to generate and appreciate unique solutions (Schappelle & Sowder, 1989). Instructional lessons should be designed so that they help “students build connections by emphasizing concrete, pictorial, symbolic, and real world representations of number.” (Weber, 1996) These recommendations support the philosophy stated by the National Council of Teachers of Mathematics which states that what an individual learns is intrinsically connected to how they learn (NCTM, 1989). This philosophy is supported by research that concludes that “the use of curriculum that reflects reform recommendations (proposed by the NCTM) can have positive effects on student’s opportunities to learn.” (Gearhart et al., 1999) Students exposed to this type of curriculum perform significantly better than students exposed to more traditional methods. Other studies take this further and state that instruction which focused on exploration and class discussion of strategies resulted in students exhibiting improved number sense (Markovits & Sowder, 1994). Knowledge of rote procedures hinders students from successfully building on prior knowledge and experiences (Resnick as cited in Mack, 1990, Mack 1990). These findings have been questioned by more recent studies.

Although the constructivist approach has been beneficial to medium- and high-achieving math students, the effect has been negligible, and it some negative, to low-achieving math students. For these students, explicit teacher-led instruction and peer-assisted learning is recommended (McDougall, Ross & Jaafar, 2006).

Research shows that there is a gap between informal knowledge, applied, real-life circumstantial knowledge constructed by the individual, and knowledge of mathematical symbols and procedures (Hiebert as cited in Mack, 1990). The instructional practices recommended by the NCTM and researchers suggest that this gap is narrowed when more constructivist approaches are incorporated in curriculum and classroom practices.

Although curriculum can be designed to foster number sense, researchers urge that number sense not be seen as a series of topics that can be trained for. They also stress that teachers should concentrate on the conceptual development required to complete questions concerned with number sense. Developing number sense should be integrated in mathematical activities rather than being viewed a designated subset of specifically designed activities (Greeno, 1991, Schappelle & Sowder, 1989). Although several researchers have gone on to develop specific number sense resources, they stress that the activities and problems only serve to stimulate thinking and discussion (McIntosh, Reys and Reys, 1997). Activities should be designed so that students have the opportunity to think about what they are doing, rather than only seeing specific relationships considered important by the teacher (Cobb & Merkel as cited in Greeno, 1989). Consider these points when examining the following study. This study was concerned with designing curriculum and teaching practices which enhanced student understanding of rational numbers. A control group of children were exposed to traditional teaching practices regarding rational numbers. The experimental group were immersed in a program that emphasized the meaning of rational numbers, encouraged students to develop spontaneous strategies, highlighted the differences between rational and whole numbers, and used alternate forms of visual representations, not relying solely on the standard pie chart. The two groups performed similarly on standard procedural questions however, the group exposed to the new program did significantly better on questions deemed novel (Moss, 1999). This study supports conclusions reached in a more extensive study conducted by Confrey (Confrey as cited in Moss, 1999). A more flexible approach supports the wide range of student abilities (Mcintosh, Bana & Farrell, 1995) and also supports the notion that mathematics is dynamic and creative. If one attempts to design individual instructional components of number sense in isolation of one another, it will appear as a series of disjointed entities and is unlikely to serve the needs of the students (McIntosh, Reys & Reys, 1997, Silver, 1990).

Teachers sometimes believe that the curriculum must be altered significantly to meet these objectives but it is suggested that teachers should examine how activities in their current curriculum can be adapted to contribute to the growth of number sense (Resnick, 1989). It may merely require that the question be altered so that it is more open-ended, that is, it can be solved by incorporating and a variety of strategies and/or have a more than one acceptable answer (Howden, 1989). Another suggestion states that the question may remain the same however, collaborative group work and/or class discussions may illuminate alternate approaches and connections not previously considered, examined or recognized by other students.

So if curriculum can be designed to foster number sense what is the role of automaticity. The type of automaticity encountered in the past centered on traditional “drill and kill” questions where students repeated hundreds of trials, all doing the same thing. Although these types of activities can be very mind-numbing, they do lead to automaticity. Sowder (1989) viewed automaticity in another form where students had examined so many variations of the questions, the strategies to solve them, and the contexts, that the children can immediately sense what should be attempted. In other words, their exposure to a wide variety of worthwhile tasks and strategies allows them to efficiently and expediently decide what strategy or strategies could be employed. This view of automaticity incorporates critical thinking opposed to the use of rote procedures which may involve very little mathematical thinking. McIntosh, Reys and Reys (1997) published three teacher resources on number sense that appear to adhere to this belief. Students are exposed to a wide variety of questions and strategies designed to foster number sense but also rely on a certain level of automaticity. Their materials support Sowder’s views of automaticity and are drastically different from the direction suggested by Bove (2003). She claims repetition leads to mastery of skills and concepts and that this practice is best carried out using graduated worksheets, usually out of context.

What Should Students be Encouraged to Do?

If students are to develop number sense, it is important that they recognize that mathematics is found everywhere, not just in school and that they attempt to work with numbers as much as possible outside of the formal classroom lessons (Resnick 1989). Teachers should ask students to draw from their informal knowledge, developed outside of school and ask students to put trust in their own knowledge. Students learn to trust their own knowledge when their approach is validated, they recognize that there are multiple procedures for solving problems. They can recognize this if students are asked to discuss and justify solutions in a classroom setting (Howden, 1989, Resnick, 1989, NCTM, 1989). With the appropriate curriculum and classroom support, students approach a new task, drawing from both informal and formal knowledge, assimilate new information and ultimately construct their own meaning to the concepts within that task (NCTM, 1989). This approach to learning means that teachers should also encourage students to take "shortcuts" (Hope, 1989). Consider the following mental computation question.

Example: Mentally compute 199 + 437.

Student One's Response	Student Two's Response
$200+437-1$	$\begin{array}{r} {\scriptstyle 1\;\,1\;\;} \\ 1\;9\;9 \\ +\;4\;3\;7 \\ \hline 6\;3\;6 \end{array}$
$637-1$	
636	

Shouldn't the teacher be supporting the "shortcut" or nonstandard approach used by student one? Research suggests that teachers should support these types of approaches because retention of these strategies is generally higher due to the strengthening of "conceptual networks" (Markovits & Sowder, 1994), that is, the improved ability to work flexibly with numerous concepts involving numbers and operations. In the first solution provided above, student one has chosen a nonstandard left-to-right approach that relies on rounding, adding, then choosing the correct direction to compensate for the rounding. This student has displayed an ability to work flexibly with the concepts of rounding, addition and subtraction.

Students should be encouraged to investigate relationships between numbers. In one article, the author suggested that students express the day of the month as an "incredible equation." One student was assigned the thirteenth of the month and here are the equations she produced.

$10+3$ $\quad$ $2\times 5+3$ $\quad$ $3\times 4+1$

$7\times 2-1$ $\quad$ $2\times 2\times 2\times 2-3$ $\quad$ Double 6, add 1.

Half of 26 $\quad$ $39\div 3$ $\quad$ $3+\sqrt{100}$

4^2-3

Start with 100, take half, take half, add 1, take half

$(200\div 10)-(2\times 3)-\frac{5}{5}$ (Howden, 1989)

Activities such as this one require students to examine relationships previously not considered in more traditional exercises. A similar line of questioning was recommended by another author but he also articulated that these questions would also encourage students to "look before they leap." The following questions are examples of this line of questioning.

Without computing, select the pairs of calculations that produce the same answer:

(a) 8×45 and 4×90

(b) 2.9×3.7 and 29×0.37

(c) 92×48 and 90×50 (Hope, 1989)

Examine and consider each of the proposed questions. Students may initially feel that the two multiplication questions supplied in question (a) are unrelated but with discussions they will discover the strategy of "halve and double" that makes some multiplication questions far more manageable. A similar relationship is illustrated in question (b) however, in this case one is multiplying by ten then dividing by ten to create the same product; a strategy that may not be initially considered by all students. In question (c), many students initially believe that the two products are equivalent because in the second case, the numbers have been decreased by two and then increased by two. These students believe that they are exploiting a similar property as identified in questions (a) and (b) but with further work and discussion they will discover that this is not the case.

Students should be encouraged to take risks and recognize that fundamentally linked to such behavior, one is expected to make mistakes. "Errors are part of the problem solving process, which implies that both teacher and learners need to be more tolerant of them. If no mistakes are made, then almost certainly no problem solving is taking place. Perfect performance may be reasonable criterion for evaluating algorithmic performance . . . , but it is incompatible with problem solving."

(Martinez as cited in Eggleton and Moldavan, 2001). Errors are important for they often provide insights to the teacher to what the student actually knows and how the student has constructed such knowledge. Sometimes teachers have a tendency to identify potential areas in problems where errors can be made. Surprisingly, this instructional practice of warning students does not reduce the number of errors made (Moldavan as cited in Eggleton & Moldavan, 2001). From the student's perspective, they learn from their mistakes (Eggleton & Moldavan, 2001).

Students must be encouraged to supply more than just answers (Sowder as cited in Reys and Yang, 1998). By doing this, students learn that the process is valued far more than the product and that the process should be reflected upon by themselves, their classmates, and teacher. Only with elaboration can all concerned learn from the experience.

Algorithms

As previously mentioned, learned algorithms comprise a significant portion of current curriculum. Although the use of prescribed algorithms is disproportionately high, there is still a need to develop algorithms. The question then arises, when should teachers teach or develop algorithms? Consider the following. Students who are well versed with prescribed algorithms are resistant to change, tend to be fixated on that procedure at the expense of other strategies which may be more efficient, and may start to abandon their informal insights (Markovits & Sowder, 1994, Trafton, 1989, Weber, 1996). The same is true for high-ability students. They will select more standard algorithms unless they are prompted to find alternate approaches (Reys & Yang, 1998). This research suggests that where it is possible new tasks should be attempted using prior knowledge and experience rather than applying a new algorithm. The sense-making should proceed the algorithm because if this does not occur, the students are reluctant to adopt, accept or even consider alternate approaches. Markovits (1989) recognizes that if school mathematics is very rule oriented then students are not given the opportunity to make decisions or judgements and therefore number sense is not fostered.

Communicating Mathematically Through Discussions

Communication, reasoning and justification are important aspects of learning mathematics. Students can make connections and improve their own understanding by listening to the strategies employed by other students and their rationale for doing so (NCTM, 1989, 2000). Students should be asked to explain and justify the strategy they use to complete an item. This might reveal to classmates some interesting and creative lines of thinking (McIntosh, Reys & Reys, 1997) and make them aware of various ways of manipulating quantities rather than just symbols (Trafton, 1989). Examining these alternate strategies also allows students to judge their ease and efficiency so that they can make better choices when encountering future questions (Sowder, 1990).

So how do teachers create situations where students communicate effectively? In an article titled "Fractions Attack!" (Alcaro, Alston & Katims, 2000), the authors examined the role of the teacher when trying to have students think and communicate mathematically. The authors provided the following suggestions.

- Have the students work together and respect one another's ideas.
- The teacher can use the simple technique of saying "I don't understand" or "I'm confused" to elicit more complete explanations from students.
- When students are providing explanations, do not make assumptions. Insist that students clarify their statements when it is appropriate.
- Ensure that all students understand the explanations provided by other classmates.

If number sense is 'making sense what number is about', it is necessary to spend considerable time discussing answers and strategies used. Questions such as these can elicit the desired discussions.

- How did you get your answer?
- What prompted you to use that approach?
- Why does this make sense?
- Can you explain it another way?
- Did anyone do it differently or obtain another answer?
- How are these ideas related?
- How does this relate to work done on other days or from other units?

It is also important to share wrong answers to identify faulty reasoning, issues regarding the wording of the question, computational errors, or other problems which might arise (McIntosh, Reys, Reys & Hope, 1995, NCTM, 2000). Teachers should also be prepared to model such behavior by sharing with their students the thinking strategies they employed when solving questions (McIntosh, Reys & Reys, 1997).

Through the course of discussions if numerous strategies are presented, should the teacher advocate one approach at the expense of another? If a teacher advocates one particular approach, then the open-endness of these types of problems are lost. In addition to this, the teacher will not be recognizing that there are different levels of thinking often associated with these different approaches. Talented and less talented students need to participate in and experience success with these activities. Stressing one strategy at the expense of another valid strategy will serve to alienate students and hinder creative thought. In many cases, teachers will find different methods that are equally as efficient and these matters should be discussed in class. Consider the following two responses to this mental computation question.

Question: Mentally compute 7×28.

Response 1:	Response 2:
7×28	7×28
$7 \times (25 + 3)$	$7 \times (30 - 2)$
$7 \times 25 + 7 \times 3$	$7 \times 30 - 7 \times 2$
$175 + 21$	$210 - 14$
196	196

Both respondents used the distributive property but in different but equally efficient manners (Sowder, 1990).

Some teachers are concerned with the class time required to discuss the multitude of strategies and ideas proposed by students. Balancing the needs of students to express their ideas with the goal of helping students learn mathematics requires the teacher to focus the discussion to address the key concepts of the day’s lesson. This insures that discussions are productive mathematically (Sherin, 2000). Teachers should also be aware that discussions provide the teachers with insights into how the students acquire understanding of a particular topic and in turn allow teachers to implement appropriate strategies and materials to facilitate learning (Forrester, 1998). Although discussions can be time consuming, they are a valuable component to the learning process.

Context Versus No Context

Context influences performance as well as the thinking strategies employed. Consider the following example where a 12 year old street vendor in Brazil was questioned about the cost of 10 coconuts and which led to a solution that differed drastically from the researcher's expectations.

Customer :	How much is one coconut?
Vendor:	35
Customer:	I'd like ten. How much is that?
Vendor:	(Pause) Three will be 105; with three more that will be 210.(Pause) I need four more. That is ... (Pause)315... I think it is 350.

One might expect that the vendor would simply add a 0 to the 35 but the vendor appeared to apply his/her knowledge of the value of three coconuts to address the problem (Carraher, Carraher & Schliemann as cited in Greeno, 1991). Other studies report that children are more likely to employ an algorithm when trying to solve $2.39 + 0.99$, opposed to $\$2.39 + \$0.99$ (McIntosh, Reys & Reys, 1997). The context of money allows students to access strategies from their informal knowledge because the calculation is done with a purpose and is representative of strategies they use in their daily lives. If teachers want the students to consider nonstandard approaches it may be facilitated with problems that are situated in real-world contexts (Hope, 1989).

Another interesting finding was discovered regarding number sense and context. Merely changing the numbers in a word problem can have a significant effect on the choice of operation selected by the student. Consider the results from these two questions given to 12 and 13 year old students in Belfast (Greer, 1987).

A train did a journey of 98.2 miles at a speed of 34.7 miles per hour. Write down the calculation you would do to work out how long the journey took in hours. Correct (Division): 57% Multiplication: 16%	A train did a journey of 25.6 miles at a speed of 37.8 miles per hour. Write down the calculation you would do to work out how long the journey took in hours. Correct: 20% Reversed the Division: 31% Multiplication: 16%

Many of the same individuals who had successfully completed the first question were unable to correctly solve the second question. Although number sense can be fostered using contextual problem (Hope, 1989), problems can arise based on the numbers used.

Other research cautions teachers in their selection of contexts. Unfamiliar contexts, contexts that require elaborate written descriptions, or contexts that include irrelevant information can cause great difficulties for some learners and ultimately impede the development of number sense. The researcher makes the following recommendations.

> This study strongly suggests that learners actively interpret information within a framework based on their own experiences. Teachers therefore need to get to know their students; to find out where they have come from, and what their goals, interests, and aspirations are. This information allows us to select a diversity of contexts which have meaning for the students and which make connections with their prior learning as well as stimulate new connections. . . Cooperative group work may go some way to overcoming contextual barriers in that students who are familiar with a particular problem context can assist others in the group to make sense of it. (p. 75, Helme, 1995)

Closing

Research recommends that instruction should tap into students' informal and formal knowledge, allow students to invent strategies, and, in some cases, stress conceptual meaning. This type of instruction allows students to view mathematics as a dynamic and creative discipline and in turn fosters the flexible thinking skills necessary to develop number sense. Students should be given opportunities to communicate mathematically where they discuss the merits of alternate approaches and share wrong answers in an attempt to identify faulty reasoning. The gradual development of number sense relies on the "doing of mathematics" (Howden, 1989) where students make sense of mathematics rather than becoming competent with a new set of algorithms specifically designed for number sense activities. This last statement is best encapsulated in the following quote. We should not be asking, "What do we expect a student with number sense to be able to do given a task?" but rather "What do we expect the student to undo?" (Markovits, 1989).

The next four sections of this paper examine the four major components of number sense; judging number magnitude, mental computation, computational estimation, and judging the reasonableness of answers.

Number Magnitude

Understanding number magnitude means that individuals should be able to compare numbers such that they can order the numbers, recognize which of two numbers is closer to a third, and to identify numbers between two given numbers (Markovits & Sowder, 1994). Comprehending that there are numbers between other numbers is an important aspect of understanding number magnitude. Although most students are comfortable working in the domain of whole numbers, difficulties arise for many students when working with decimals, fractions, and/or percents. Even though students recognize that there is space between 1 cm and 2 cm on a ruler then don't connect that space with number (Schappelle & Sowder, 1989). Although judging number magnitude includes whole numbers, integers, and rational numbers (decimals and fractions), much of the research as been focused on rational numbers. For this reason, this paper will examine the issue of number magnitude with respect to rational numbers.

Most students do not make the connection between their understanding of fractions and their understanding of decimal numbers (Hiebert, 1984, Markvovits & Sowder, 1991, Reys &Yang, 1998). When ordering numbers, many students initially separated fractions from decimals because they failed to recognize how these numbers are related to each other (Markovits & Sowder, 1994). In a study previously mentioned, 69% of education students were unable to order the numbers $0.53, \frac{14}{13}, \frac{5}{12}$ and 0.993 correctly (Markovits, 1989). Even the language used by teachers and students to describe decimals and fractions fails to emphasize the relationship between these proportional quantities. In one study students were asked to express 0.4 in more than one way. Nearly all third and fourth graders in this particular study were able to respond "zero point four" however, only one of the 35 students were able to supply the answer "four tenths." The results only improved slightly when the students were given choices for alternate ways to express the decimal. Only one of the fifteen third graders and three of the remaining 19 fourth graders were able to correctly identify "four tenths." The only student who correctly answered the question without prompting claimed knowledge from outside the school setting, specifically by watching and listening to time keeping at basketball games where they monitor tenths of seconds (Glasgow, Ragan, Fields, Reys & Wasman 2000).

Other research suggests that students also fail to see how percent is linked to decimals and fractions. This is not surprising since many textbooks teach them in isolation of one another with only cursory discussions of conversion (Sweeney & Quinn, 2000). Markovits and Sowder (1994) take this further to say that when they do offer opportunities for comparison of decimals, fractions and percents, the textbooks often provide procedures that do not call on any rational number sense.

If this problem exists, how can it be rectified? Curriculum, teacher resources and instructional practices should be modified or changed so that emphasis is placed on discovering and understanding the relationship among different ways of representing proportional quantities (Bay, 2001, Moss, 1999, Markovits & Sowder, 1994, Sweeney & Quinn, 2000). As previously mentioned, students must be given the opportunity to explore, make connections, and build on previous knowledge (NCTM, 2000). These modifications should also allow students to:

(1) learn and work with decimals, fractions, and percents simultaneously,
(2) use more intuitive approaches to understand the relationship,
(3) use benchmark values to first develop an understanding of the relationship,
(4) use pictorial representations and manipulatives that were traditionally only used when working with fractions, and
(5) use the appropriate mathematical language.

Most resources suggest using fractions to lead into decimals and then following up with percents. One researcher suggested that the introduction to the rational number domain should be in the reverse order. She cited several reasons for doing so.

(1) By the age of 10, many children have well-developed intuitions regarding proportions and the same is true regarding numbers from 1 to 100. Percents are a natural extension of these two well-established intuitions.
(2) The students in this study used Macintosh computers in their classrooms. When files are being transferred on these types of computers, a “number ribbon” appears continually updating what percentage of the file has been transferred. Since the students were familiar with this representation, it seemed like a logical place to start.
(3) By initially working with percents, they postponed the problem of having to work with different denominators as is the case with most decimals and fractions.

(4) Every percentage value has a corresponding fractional and decimal equivalent that is easy to determine. The converse is not true because the conversions can be conceptually difficult.

(Moss, 1999)

In the study, one group of students was taught using traditional practices while the other group was taught using this newly developed curriculum which relied on more conceptual development versus procedural development. The researcher found that the two groups performed equally well on traditional questions however, the group using the newly developed materials significantly outperformed the first group on novel questions. Unfortunately, the study didn't go further to examine what effect the new order (percent, decimals, then fractions) had on the results. This could have been established if a third group of students were subjected to curriculum that also stressed conceptual development however using the order (fractions, decimals, then percents) typically seen in most resources.

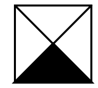
Prior to initiating activities on number magnitude, one researcher suggested that 'warm-up' activities comprised of questions similar to the ones below should be done.

Are 1.7 and $\frac{1}{7}$ the same or are they different? Why?

Are 0.5 and $\frac{6}{12}$ the same or are they different? Why?

Questions such as these provide opportunities for rich classroom discussions and foster understanding of rational numbers (Sowder, 1995). Teachers should encourage students to consider using pictorial representations or manipulatives to support their answers (Glasgow, Ragan, Fields, Reys, and Wasman, 2000).

Another 'warm-up' activity was recommended by two other researchers. They proposed that benchmark quantities such as 25%, 80%, 100%, 0.4, 0.75, $\frac{1}{5}$, $\frac{1}{3}$, and $\frac{3}{5}$ could be used to create activities which require students to examine relationships between decimals, fractions and percents. Students would be broken into eight groups and each group would be responsible for creating four cards, each providing a different representation of a particular benchmark quantity. For example, the first group would be given 25% and produce the following four cards.

25%	0.25	$\frac{1}{4}$	

Each group would hand in their four cards, such that the teacher now has 32 cards, four different representations of each of the eight benchmark quantities. These cards would be reproduced and then each group would receive 32 cards. The students turn the cards face down on a desk and shuffle them and then individually turn two over at a time, trying to obtain a match. Students who participated in activities such as these displayed a better understanding of the relationship between fractions, decimals and percents. Prior to such activities, students described fractions as “one number on top of another with a line between them” and decimals as merely a “dot.” Upon completion students considered fractions to be “part of something” or “a portion of a whole number” and decimals as the “same thing as a fraction” just “written in a different way.” (Sweeny & Quinn, 2000)

One author recommended that prior to doing formal exercises from textbooks or worksheets on number magnitude that a class activity involving a large rope, which represents a number line, should be used. They recommend that students initially work with whole numbers. The endpoints of the rope are defined and then students randomly draw a card and are asked to position themselves along the rope. They are then asked to justify the position they selected. This activity is then altered so that students are eventually working with decimals and fractions simultaneously (Bay, 2001).

It is recommended that formal number magnitude activities could now be initiated. These activities would include questions which ask students to:

(1) put numbers in ascending order,

Example: Order from smallest to largest $0.65, \frac{7}{5}, 0.07, \frac{3}{7}$

(2) identify which of two numbers is closer to a third, and

Example: Is $\frac{3}{8}$ or $\frac{6}{11}$ closer to $\frac{1}{2}$?

(3) identify numbers between two given numbers.

Example: Find a fraction between 0.6 and 0.7.

(Markovits & Sowder, 1994).

As previously mentioned, obtaining correct answers to these types of questions is not the sole objective. The sharing of ideas and strategies is a significant component to fostering number sense. Recognizing and understanding an alternate strategy is mathematically enriching (NCTM, 1989, 2000). Class discussions are recommended. Consider the strategies employed by high-level eighth grade students who determined that there were an infinite number of fractions between $\frac{2}{5}$ and $\frac{3}{5}$.

Strategy One – Express the Numerator as a Decimal

$$\frac{2.1}{5}=\frac{21}{50},\ \frac{2.2}{5}=\frac{22}{50},\ \frac{2.3}{5}=\frac{23}{50}\ ,\ \ldots\ ,\ \frac{2.9}{5}=\frac{29}{50}$$

Strategy Two - Express the Original Fractions with a Denominator other than 5

$$\frac{2}{5}=\frac{400}{1000}\ \text{ and }\ \frac{3}{5}=\frac{600}{1000}$$

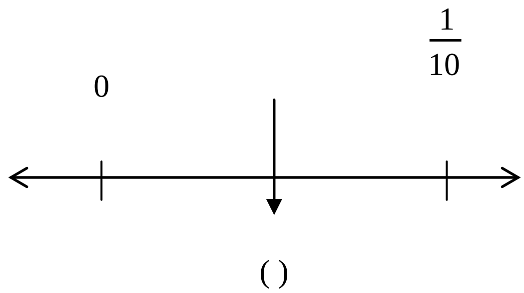
Therefore the following fractions are reasonable.

$$\frac{401}{1000},\frac{402}{1000},\frac{403}{1000},\ldots,\frac{599}{1000}$$

One middle-level student converted the fractions to 0.4 and 0.6 and realized that the value 0.5 was between them but was unable to convert it to a fraction that made sense to him/her given the numbers provided in the question (Reys & Yang, 1998).

A similar problem was examined by the same two researchers in another article. Students were given the following multiple choice question and asked to justify their selection.

Which of the following fractions best represents the value of ( )?



(a) $\frac{5}{10}$ (b) $\frac{5}{100}$ (c) $\frac{1}{100}$ (d) $\frac{5}{1000}$

Why? Please explain and justify your response.

Initially students volunteered answers however most of them were incorrect. Many students incorrectly selected $\frac{5}{10}$ because they had considered the endpoint to be 1 rather than $\frac{1}{10}$. The teacher then decided to break the students into groups and work collaboratively on the question. The groups were then asked to present and defend their answers. Each group was able to obtain the correct answer however four distinct approaches were identified.

(1) Using Equivalent Fractions

Student Response: "We changed $\frac{1}{10}$ to $\frac{10}{100}$ and then cut it in half, so the answer is $\frac{5}{100}$."

(2) Converting to Decimals

Student Response: "We changed $\frac{1}{10}$ to 0.1, then half of 0.1 is 0.05, and 0.05 is equal to $\frac{5}{100}$."

(3) Working with Decimals and Fractions

Student Response: “Half of $\frac{1}{10}$ is $\frac{0.5}{10}$. We changed the 0.5 to be an integer and $\left\lfloor \frac{0.5}{10} \right\rfloor$ became $\frac{5}{100}$.

(4) Working Backwards

Student Response: “$\frac{5}{10}$ is over $\frac{1}{10}$ so that can’t be the answer. The third answer is $\frac{1}{100}$ but that is too small. The fourth answer, $\frac{5}{1000}$ is smaller than $\frac{1}{100}$. We agree that the answer is the second one.”

By leaving the question open-ended, students were able to select the representation, decimal, fractions or a combination of both, that made the most sense for them. This activity and how it was addressed by the teacher demonstrates how an enriching learning experience can be fostered by encouraging classroom discussion where all students and strategies are valued (Yang and Reys, 2001).

If the previous examples represent the types of desired responses teachers want to illicit from their students, what types of results were generally being exhibited by students? On pretests, many students were unable to identify that there are an infinite number of decimals or fraction between 0.74 and 0.75 and the same was true when trying to identify values between $\frac{2}{7}$ and $\frac{3}{7}$ (Markovits & Sowder, 1994). In another study, six of the eight middle-level students believed that there were 9 or 10 different decimals between 1.42 and 1.43. They stated the values 1.421, 1.422, 1,423, ..., 1,429. In that same study, all of middle-level students believed that there were no numbers between $\frac{2}{5}$ and $\frac{3}{5}$ due to the common belief that $\frac{3}{5}$ followed $\frac{2}{5}$ (Reys & Yang, 1998). Markovits and Sowder (1994) implemented curriculum that was more conceptually rather than procedurally oriented and found that number sense improved significantly. In the domain of number magnitude, students were now able to order numbers and identify correctly which of two numbers was closer to a third however, students still had difficulties locating fractions between other fractions. Reys and Yang (1998) were not concerned with testing new curriculum but rather examining if there was a relationship between pencil-and-paper computational skills and non-computational approaches which relied on number sense. They found that high skill in written computation was not necessarily accompanied by number sense. They concluded that these students, who were comfortable with algorithms involving decimals and fractions, lacked the necessary conceptual understanding to deal with these more novel questions such as finding a number between two other numbers.

Several issues regarding fractions, decimal, and percents may arise when proceeding with number magnitude activities and their related discussions. Many of these important issues have been addressed in several studies. Let's first examine issues concerning decimal numbers. Results from different studies make it difficult to reach a definitive conclusion regarding the level of expertise students have with decimals. One study reported that most students successfully ordered decimal numbers in the pretest component of the study (Markovits & Sowder, 1994). One might infer from this that these students had a conceptual understanding of decimal numbers however, this might not be the case. In another study, the researcher reported that when students are having difficulties comparing 0.45 and 0.6, they are sometimes told to 'add a zero' to the 0.6 to make it into 0.60. Students are often able to successfully complete the question now but has this strategy fostered number sense? If these students are successfully arranging decimals based on a prescribed rule, then a true understanding of decimals is lacking. This researcher proposed that instead, one should ask the students, "Which is larger 45 one-hundredths or 6 tenths?" (Sowder, 1995).

The results from the Third International Mathematics and Science Study reveal that most students do poorly on questions dealing with decimals, yet comparatively better on some questions involving fractions (Glasgow, Ragan, Fields, Reys & Wasman, 2000). Another researcher's study identified several common misconceptions regarding decimal numbers that have been stated in the following list.

Misconceptions about Decimals

- Longer decimal numbers are of larger value.
- Longer decimal numbers are of smaller value.
- Putting a zero at the end of a decimal number makes it ten times as large.
- If you do one thing on one side of a decimal, you must do the same thing on the other side of the decimal. (example: 3.4 + 1 = 4.5)
- Decimals are below zero (i.e. negative numbers).
- Place-value columns include "oneths" to the right of the decimal point.

- One-hundredth is written as 0.100.
- $\frac{1}{4}$ can be written as either 0.4 or 0.25.

(Irwin, 2001)

As previously mentioned, research shows that identifying potential errors to students is not recommended. Instead, address the concerns as they arise in class discussions. Irwin (2001) also found that students who worked with contextual problems made significantly more progress in their knowledge of decimals opposed to students who worked with noncontextual problems. Based on the results from most of these studies, one can conclude that some students still lack some level of conceptual understanding of decimal numbers.

In regards to working with percents, traditional instruction in their application that is less-intuitive and rule-driven, narrows the strategies employed by students when working with percents. This was demonstrated in two ways. In the first case, grade 5 students performed equally as well on problems that were contextual and noncontextual however, students in grade 9 struggled with many of the contextual problems yet performed well on noncontextual problems. These two grade levels also differed greatly on their choices of strategies. The grade 5 student who had significantly less formal instruction on percents, relied on strategies which relied on more intuitive approaches that often involved using benchmark values (10%, 25%, 33% and 75%). The older students, who ultimately didn't perform as well, relied more on extensively practiced algorithms. However, students in grade 9 did use benchmarks but primarily as a means to check answers derived using other strategies (Lembke & Reys, 1994).

Two distinct problems typically arise when students are asked to address number magnitude questions which involve fractions.

(1) Students often make the mistake that $\frac{5}{6}$ and $\frac{9}{10}$ are equivalent because they believe that each is "one piece away" from one (Markovits & Sowder, 1994).

(2) Children are susceptible to treating the numerators and denominators of fractions as separate entities (Kerslake as cited in Markovits & Sowder, 1994).

Again research suggests that these issues should be addressed in class discussions as they arise versus providing warning of these potential errors.

In closing, research into rational numbers demonstrates that teaching practices which are beneficial to student understanding of number magnitude:

(1) ask students to construct their own meaning using informal and formal knowledge,
(2) are conceptually based,
(3) ask students to discuss the strategies they used,
(4) are receptive to a wide range of strategies,
(5) and work with different representations of rational number.

Mental Computation

Mental computation is the process of calculating the exact numerical answer without the aid of any external calculating or recording device. Research shows that as adults over 80% of the mathematics we encounter in our daily lives involves the mental manipulation of numerical quantities rather than the traditional paper and pencil computations so often stressed in elementary and middle school classrooms (Edwards as cited in Reys & Reys, 1995). Not surprisingly then most students feel that mental computation was important (McIntosh, Bana & Farrell, 1995, Reys & Reys, 1995) however, they believe that written computation is learned in school while mental computation is learned outside of school (McIntosh, Bana & Farrell, 1995). This may explain why the results for mental computation differed for contextual problems opposed to noncontextual problems. Students see contextual problems as 'outside of school' problems. Consider this example where performance and the strategies employed differ based on the context of the question even though mathematically they are seemingly comparable tasks.

Question 1: Bananas cost $1.20 per kilogram. Apples cost $1.70 per kilogram. If you purchase 3 kilograms of bananas and 0.5 kilograms of apples, how much would it cost?

Question 2: $1.2 \times 3 + 0.5 \times 1.7$

The first question is contextual. The second question is noncontextual but involves using the same numbers and operations. The researcher found that the students performed significantly better on the first question and were more likely to employ mental math strategies for the first question opposed to paper-and-pencil strategies for the second. (Hope, 1989).

Although students may value mental computation, they may not be able to perform even the most straightforward calculations mentally. Consider that on the Third National Mathematics Assessment, only 45% of 17 year olds were able to multiple 90 and 70 mentally (cited by Hope & Sherrill, 1987). The findings made by McIntosh and his colleagues and those made by Hope and Sherrill may indicate that little classroom time has been devoted to doing mental computation.

A study regarding performance on mental computation of students in grades 2, 4, 6, and 8 showed the range of strategies selected by students was very narrow and that the most popular strategy selected by grade 4 and grade 8 students reflected the learned paper/pencil strategy. Many students were even unable to propose an alternate strategy when prompted and even surprised that there were alternate strategies (Reys & Reys, 1995).

If mental computation is to be incorporated into teaching practices, how and when should it be done? Formal written algorithms are often mastered before some teachers are prepared to consider mental computation. Sowder (1990) disagrees with this philosophy and, as previously mentioned, she is supported by other researchers of number sense who recognize that students become fixated on these algorithms at the expense of alternate strategies. Mental computation activities should be pervasive in the curriculum and the teacher should focus on discussions of how the problems were solved rather than concentrating on mental computation drill (NCTM, 1989, 2000, Schappelle & Sowder, 1989). It may not be desirable to teach specific mental computation strategies, rather allow students to explore, discover and share strategies best suited for their needs and abilities (McIntosh, Bana & Farrell, 1995). Children are able to produce a wide variety of efficient strategies even though they may have had little direct teaching of algorithms (Carol as cited in Heirdsfield, 2000). Both of these statements are supported by research on highly skilled mental calculators who had discovered effective and efficient strategies by merely "playing with numbers" (Hope, 1987).

What types of responses to mental math questions might a teacher expect? Research shows that teachers can expect a wide range, varying from very traditional use of algorithms to very creative strategies which indicate a high-level of number sense. In one particular study, the researchers organized student responses into one of the four categories.

Categories:

(1) *Standard:* The student uses techniques that model previously taught pencil-and-paper algorithms.

(2) *Transitional:* The student is still bound to the algorithm however, they are paying more attention to the numbers involved and less to the procedure advocated by using the algorithm.

(3) *Nonstandard with no reformulation:* The student took a novel approach however the numbers were not reformulated.

(4) *Nonstandard with reformulation:* The student took a novel approach however the numbers were reformulated to make the computations easier.

(Markovits & Sowder, 1994).

Consider the following example and the three strategies proposed by students.

Example: Mentally compute 7×28.

Student 1	Student 2	Student 3
"7 twenties is 140; 7 eights is 56; together that's 196."	"28 is 4 times 7; 7 squared is 49; 4 times 50 is 200, then 4 times 49 is 4 less than 200; that gives 196."	"7 times 8 is 56: put down the 6, carry the 5; 7 times 2 is 14, add the 5, so 19 next to the 6 is 196."

The first two students have used nonstandard approaches that are more efficient and rely less on short-term memory opposed to the algorithmic approach used by the third student. According to the categories proposed by Markovits and Sowder, the first student's strategy is nonstandard with no reformulation, the second student's strategy is nonstandard with reformulation and the third student's strategy is standard.

Most unskilled students used a digit-by-digit, right-to left process to do even some of the more straightforward mental calculations. In one case a student took 34 seconds to complete the question 20×30 because she relied on the algorithms taught in class. She explained,"30 is on the top, and 20 is on the bottom. 0 times 0 is 0; 0 times 3 is 0. Put down a 0. And 2 times 0 is 0, and 2 times 3 is 6. And then you add them together, and you'd get ... 600?" (Hope & Sherrill, 1987) This approach led to the excessive use of the 'carry' operation. In the same study, researchers learned that skilled students selected strategies that reduced the number of 'carries' and reduced the burden on the short term memory. For example, one skilled student used no 'carry' operations when solving 15×16. The child stated, "80 and 16, move one over, 160. And, 160 and 80 is 200 and 40 more, which equals 240." The burden of the 'carry' with questions such as 25×48 is so excessive that performance suffers. Skilled students would often use the half and double strategy with this type of question.

Example: 25×48

50×24

100×12

1200

Other skilled students may use their knowledge that "4 times 25 is 100."

Example: 25×48

$25 \times 4 \times 12$

100×12

1200

Other students may use the distributive property to successfully complete the question.

Example: 25×48

$25 \times (50 - 2)$

1250-50

1200

While other students may attempt to create a contextual situation to complete the problem.

Example: 25×48

"I wanted to work with money. I wanted to know how much money I would have if I had 48 quarters. If 4 quarters are worth $1, then 48 quarters is worth $12 or 1200¢. The answer is 1200."

These are just four possible strategies that students might employ. Note that these strategies would be classified as nonstandard with reformulation.

Students persist on using standard algorithms when mentally adding numbers that are not large. However, as the values got larger many adopted the nonstandard process of rounding, adding, and compensating for the rounding (Markovits & Sowder, 1994).

Example:

$157 + 99$

$157 + 100 - 1$

$257 - 1$

256

A similar nonstandard approach for subtraction is used by many students, unfortunately, students have difficulties choosing the correct direction for compensating for the rounding (Markovits & Sowder, 1994).

Examples:

$348 - 99$	$348 - 99$
$348 - 100 + 1$	$348 - 100 - 1$
$248 + 1$	$248 - 1$
249	247
correct	incorrect

Students should be encouraged to examine the entire question before considering a strategy. Consider the following problem.

Example: Do $47 + 26 + 18 - 26$ in your head.

This question can be successfully completed by merely working from left to right. However this strategy is much more difficult. The child using number sense examines the entire problem and recognizes that there is a relationship that can be exploited such that the student is only required to add 47 and 18.

In one study, researchers examined the mental computation skills of a gifted 13 year old female identified as Charlene. Her parents had not noticed her exceptional gift until she was about 10 years old and to the best of their knowledge she had not been taught any of these mental computation techniques. She was self-taught and had discovered numerous strategies by merely playing with numbers. She did possess two unique abilities that greatly influenced her ability to do mental computations. She had an incredible short-term memory and she appeared to just know the squares of all two-digit numbers and a significant number of three-digit numbers (Hope, 1987). Some of the strategies she employed were very similar to ones previously viewed by other researchers with other students. Examples of these can be found below.

Charlene's More Routine Strategies:

(1) Using the Distributive Property

Additive Distribution	Subtractive Distribution
16×72	17×99
$16 \times (70 + 2)$	$17 \times (100 - 1)$
$1120 + 32$	$1700 - 17$
1152	1683

Difference of Squares: $(a - b)(a + b) = a^2 - b^2$

49×51	13×17
$(50 - 1)(50 + 1)$	$(15 - 2)(15 + 2)$
$50^2 - 1^2$	$15^2 - 2^2$
$2500 - 1$	$225 - 4$
2499	221

(2) By Factoring

18×72	25×48	50×64
$18 \times (18 \times 4)$	$\frac{100}{4} \times 48$	100×32
$18^2 \times 4$	$100 \times \frac{48}{4}$	3200
324×4	100×12	
1296	1200	

(3) Recall (" I just know that one.")

Other strategies were far more difficult and they relied far more on her exceptional short-term memory and extensive knowledge of squares. Examples of these more rigorous strategies are supplied below.

(1) What is the value of 26^2?

She forgot but used the following strategy.

She knew:

$4^2 - 3^2 = 3 + 3 + 1$

$5^2 - 4^2 = 4 + 4 + 1$

$6^2 - 5^2 = 5 + 5 + 1$

so

$(a+1)^2 - a^2 = a + a + 1$

or

$(a+1)^2 = a^2 + (a + a + 1)$

Therefore

$(a+1)^2 = a^2 + (a + a + 1)$

$26^2 = 25^2 + (25 + 25 + 1)$

$26^2 = 625 + 51$

$26^2 = 676$

(2) Mentally compute 87×23.

Charlene's Response:

87×23

$(29 \times 3) \times 23$

$29 \times (3 \times 23)$

29×69

$69 \times (30 - 1)$

$69 \times 30 - 69$

$2070 - 69$

2001

(3) Mentally compute 456×123.

Charlene's Response:

456×123

$\frac{456}{3} \times \frac{123}{3} \times 9$

$152 \times 41 \times 9$

$19 \times 8 \times 41 \times 9$

$19 \times 41 \times 8 \times 9$

$(20 - 1) \times 41 \times 8 \times 9$

$(820 - 41) \times 8 \times 9$

$779 \times 8 \times 9$

$(700 + 70 + 9) \times 8 \times 9$

$(5600 + 560 + 72) \times 9$

6232×9

$(6200 + 32) \times 9$

$55800 + 288$

56088

These examples provide one with an appreciation for the diversity of nonstandard approaches that may be employed by students.

Mental computation should not be limited to whole numbers, working with rational numbers should be a natural extension. (Sowder, 1995). The questions should create opportunities for students to understand the interconnectiveness of different representations of rational numbers in a similar manner advocated in the section on number magnitude. Consider the three questions below.

(a) $0.5 + 0.75$ (b) $\frac{1}{2} + \frac{3}{4}$ (c) $\frac{1}{2} + 0.75$

Research has shown that most students do not have difficulties dealing with question (a), however that same group of individuals is likely to get question (b) and/or (c) wrong. The students don't recognize that its the same question. They view decimals and fractions as separate entities (Reys & Reys, 1995).

Mental computation is cognitively demanding because it requires the subject to hold interim calculations in memory while simultaneously trying to retrieve and use different facts and strategies (Heirdsfield, 2000). Surprisingly there is only a weak correlation between mental computation performance and short-term memory capacity. Researchers believe that many of the nonstandard approaches alleviate the need to have an exceptional short-term memory. However, the recall of larger numerical equivalents such as the squares of some two digits number do have a bearing on the ability of a child to correctly complete mental computations (Hope & Sherrill, 1987).

There was one significant discrepancy in the research regarding mental computation. This discrepancy was concerned with the method by which the question was presented. One study showed that there was no significant difference in results between oral and visual presentation of mental computation questions at any grade level (McIntosh, Bana & Farrell, 1995). However, another study reported that mental computation results improve significantly when the items are presented visually opposed merely to being presented orally (Reys & Reys, 1995). When the item 182 + 97 was presented orally to grade 6 students only 47% responded correctly. However, when the same item was presented visually, the success rate rose to 80% (MacIntosh, Reys & Reys, 1997). However, all of these researchers agree that students should be exposed to both oral and visual methods of presentation.

In closing, students can and do formulate their own strategies for doing mental computation. Although these strategies are not always correct, the research demonstrates that they are more accurate and show greater number sense than teacher taught strategies (Heirdsfield, 2000). Although students who showed a preference to use mental computation were generally more able mathematics students (McIntosh, Bana & Farrell, 1995), this can be changed if students are exposed to activities that foster number sense. Researchers observed that these students still use familiar standard algorithms when attempting to solve easy questions mentally however, they were far more likely to use nonstandard approaches when dealing with more challenging questions such as 24×25, $475 \div 25$, and $76 + 53 + 17 - 53$ (Markovits & Sowder, 1994).

Computational Estimation

Computational estimation refers to one's ability to estimate answers to numerical computations. Why teach computational estimation? With today's technology, why can't exact calculations be used instead of estimations? In answering these questions Silver (1990) argues that there are situations where estimation is desired.

(1) Prior to engaging in an exact calculation so that one might be able to identify error or select a method of exact computation.

(2) When judging the reasonableness of an exact computation.

(3) As an alternative to exact computations where the numbers are unknown or impossible to know exactly. For example, if one wanted to know how much food a whale consumes or how much spending money one should budget for a trip, an estimation is best suited.

Estimation is a more difficult task for young students than teachers realize (Sowder, 1995). Some researchers advise that instruction on number magnitude and mental computation should precede instruction on computational estimation because many of the concepts and strategies employed in these other two categories of number sense are also utilized when doing computational estimation (Case & Sowder as cited in Markovits & Sowder, 1994). If students lack the exposure to and experience with these different strategies and concepts then students may have problems with computational estimation. Researchers also caution designers of curriculum that some students are not developmentally ready to estimate. In one study, the seriousness of the difficulties experienced by younger children prompted the researchers to suggest that maybe mental computation should not be initiated until students were in middle school (Sowder & Wheeler, 1989). If estimation is introduced too early, then the strategies become rules and estimation does not serve to foster number sense (Schappelle & Sowder, 1989). For this reason, some researchers have argued against teaching specific estimation strategies because they fear that such teaching would only lead to a new set of algorithms, in this case, estimation algorithms (Reys, B. J., 1989). Instead of teaching specific estimation algorithms researchers suggest that students be encouraged to solve estimation problems using intuition (Carpenter, Coburn, Reys & Wilson, 1976).

One of the questions that arises, for students, as well as teachers, is "What is a good estimate? This is a complex issue. The research shows that, for students, two sources of validation arise. The first source is the teacher. If the teacher thinks the estimate is good, the student thinks it is good. Students believe that since the teacher knows the exact answer, the teacher can judge how appropriate the estimate is (Forrester & Pike, 1998). A second way students determine if an estimate is good is by following up the estimate with calculating the exact answer themselves. Students then judge how close their estimate is to the exact answer (Carpenter, Coburn, Reys & Wilson, 1976, Sowder and Wheeler, 1989). Research, however, does not specify what criteria student's use to judge "closeness." This technique of validation requires that every estimation should be followed by an exact calculation, which ultimately defeats the purpose for the estimation. Why would one bother with an estimation if you are always end up working out the exact answer? Although these are the two sources of validation typically used by students, they can be counter-productive. Alternate forms of validation should be endorsed by the teacher.

Teachers who are engaged in developing number sense find that the comparison of the exact answer to the estimated answer, regardless of the criteria used, is not sufficient for determining how "good" the estimate is. Indeed, one research project (Hope, 1989), found that when teachers fixate on the exact answer, it can be counter-productive to recognizing good estimating. Focussing on the exact answer may result in instructors requiring an unrealistic degree of precision in an estimate. In contrast, an estimate should be judged for its reasonableness, at least in part, by considering the process used to obtain it (Carpenter, Coburn, Reys and Wilson, 1976). Proximity alone, does not determine "good."

Several strategies have been identified to help students make, and recognize when they have, good estimates. One approach is that, rather than comparing results with the exact answer, students should be encouraged to refine their initial estimate (NCTM, 1989). This allows students to reflect upon the strategy and numbers employed to complete the estimation. The student may decide to refine or change the strategies or numbers used. They can then compare their new estimate with the initial estimate. A second recommended strategy is to make it clear to students that the *process* of estimating is valued. Through class discussions, students can learn that the process is under scrutiny, rather than the answer itself. If they share and examine the strategies employed by other students they can judge how reasonable their own strategy and estimate are.

Although the NCTM document recommends that teachers and students avoid calculating exact answers, they also suggest using a particular estimation game that appears to be in conflict with that original recommendation. In this game, students turn over a card that reveals the price of an item and a particular discount in percent. Each student is then asked to estimate the final, discounted price. The winners of the various rounds are determined based on how close the student is to the exact answer. This is in direct conflict with the approach recommended earlier in the same document.

What types of problems arise when students are asked to do computational estimation? Older children resist estimating at all, preferring to calculate exact answers instead. In one study when students were asked to estimate the value of 28.93×20.987, many produced 607.15391. This precision indicates that the students did not estimate. They answered the question using an algorithm or technology. In some cases, students will calculate exact answers and round off the answer produced so that it appears to be an estimate. This is not surprising since exact computation is generally valued and rewarded in most school situations (Bove, 2003, Silver, 1990, Sowder & Wheeler, 1989, Sowder, 1995).

Older children are also reluctant to except more than one "right answer" to an estimation problem. In one study, students associated estimation with the school-learned rounding rules and therefore refused to accept estimations which did not adhere to those rounding rules explicitly (Sowder & Wheeler, 1989). A mistake of this type was found in a teacher resource. In the resource by Bove (2003) the following example was provided..

Estimate the value of 28.93×20.987

The author stated that the correct answer to this question was 609 (29×21). She failed to recognize that there are a range of acceptable estimates because she had blindly applied the rounding rules.

Many students, as well as a few teachers, believe that estimation merely involves the rounding of numbers according to prescribed rules (Schappelle & Sowder, 1989). If rounding rules are applied in the absence of number sense, some troubling student answers can be obtained. Consider the following two examples. In both cases, the students have failed to recognize that 0.53 and 0.46 are approximately equal to $\frac{1}{2}$ and have instead applied whole number rounding rules and obtained estimations which are far too large in the first case, and far too small in the second case (Sowder, 1995).

Estimate: 46×0.53 | Estimate: 122×0.46

Student Response: | Student Response:

$46 \times 1 = 46$ | $122 \times 0 = 0$

Other students apply the rounding rules to all parts of the number. In this example provided in the article by Schappelle and Sowder (1989), the student applied the rounding rules to the "whole" part and "decimal" part of the number separately.

Estimate: $152.621 + 49.23$

Student Response:

$150.6 + 50.2$

200.8

Although the estimate is reasonable, the strategy employed is flawed and shows a lack of number sense.

A similar problem can exist with younger children when they are asked to estimate the following.

Estimate: 3575 + 5876 + 347 + 8

Many students have difficulty ignoring or dropping the 8. They failed to consider dropping insignificant numbers or parts of numbers (Hanson & Hogan, 2000, Schappelle & Sowder, 1989).

Issues regarding multiplication and division often arise when students initiate computational estimation. A few students subscribe to the belief that "multiplication makes things bigger." (Graeber & Tirosh as cited in Markovits & Sowder, 1994, Reys & Yang, 1998) In one study students were asked if 72×0.46 is greater than, less than or equal to 36. The majority of students answered correctly however one grade six student responded that the answer was much greater than 36 because multiplication always results in a larger answer (Reys & Yang, 1998). A more commonly held misconception is that "division makes things smaller" (Graeber & Tirosh as cited in

Markovits & Sowder, 1994). Students often fail to recognize that dividing by a number less than one, results in a quotient larger than the original number being divided.

Examples:

Is $234 \div 0.92$ more, less than, or equal to 234?

Is $56 \div \frac{3}{11}$ more, less than, or equal to 56?

In a study of 77 college undergraduate students who were deemed to have average to above average mathematical ability, researchers found that most students did fairly well completing estimation questions involving integers with slightly better performances in the operations of addition and subtraction opposed to multiplication and division. However, many students could not even provide any reasonable estimate for problems involving fractions. They typically focused on trying to find common denominators even though it could not be done easily. They would not consider other strategies such as substituting a fraction that can be worked with easily. Some students attempted to mentally calculate common denominators but when this failed they merely added or subtracted across the numerators and denominators even though they had been drilled that "you are not supposed to" (Hanson & Hogan, 2000). In another study, grade nine students where asked to estimate $\frac{11}{12}+\frac{7}{8}$ on a multiple-choice test. The majority of the students selected the answers 19 or 20 (Moss, 1999). This implies that these students were either adding the numerator of one fraction with the denominator of the other or adding the denominators of the two fractions. This supports Kerslake's finding, which was mentioned earlier, that students treat numerators and denominators as separate entities (as cited by Markovits & Sowder).

This paper has identified several problems associated with computational estimation. These problems are summarized below.

- Some students resist computational estimation preferring to compute exact solutions.
- Some students fail to recognize that there is a range of acceptable answers opposed to one correct answer.
- Some students believe that rounding rules must be followed explicitly.
- Some students fail to consider dropping insignificant numbers or parts of numbers.

- Some students believe that “multiplication makes things bigger” and “division makes things smaller.”
- Some students have difficulties initiating or attempting computational estimation problems involving fractions.

If these are the types of problems that can arise, how can teaching practices be changed or modified so that students can successfully complete computational estimation questions? The first suggestion is that estimation should not appear as an isolated topic but rather should be integrated throughout a curriculum. Opportunities arise daily where computational estimation can be used (Carpenter, Coburn, Reys & Wilson, 1976).

The second suggestion is that students examine a variety of estimation strategies. Research has shown that there is a positive correlation between estimation performance and number of estimation strategies that students are familiar with (Harriss & Hook as cited in Hanson & Hogan, 2000). Students also recognize the need and importance for different estimation strategies as noted by Hanson and Hogan (2000). Students can examine a variety of estimation techniques if teachers promote dialogue among students where one could discuss the merits of different approaches. (Reys, B. J., 1989). Consider the following responses to this estimation problem.

Estimate: 0.16×241

Student 1	Student 2	Student 3
0.2×240	15% of 240	$\frac{1}{6} \times 240$
48	36	40

All of these approaches are acceptable for this question even though one might argue that the level and type of number sense differs between these three students. When students have the opportunity to view and discuss these alternate approaches, they have a greater potential of gaining greater insight into this topic of estimation and developing new thinking strategies. Consider this response to an estimation question that a grade nine student shared with the class. The student noticed and applied “nice” number relationships when attempting the following estimation problem (Reys, Rybolt, Bestgen & Wyatt as cited in Sowder, 1995).

Estimate: $\dfrac{347 \times 6}{43}$

Response:

$347 \times \dfrac{6}{43}$ • it's easier to divide 6 by 43 first.

$347 \times \dfrac{1}{7}$

• $\dfrac{6}{43}$ is approximately equal to $\dfrac{1}{7}$.

$350 \times \dfrac{1}{7}$

50

• round 347 to 350 because it's easy to multiply by $\dfrac{1}{7}$.

Many students may not consider this strategy unless it arose in class discussions.

The third suggestion is that students be exposed to pre-estimation problems prior to a lengthy investigation of estimation (Sowder, 1995). Many of the number magnitude and mental computation questions previously mentioned in this paper are good examples of pre-estimation problems. The following questions are examples of such pre-estimation problems.

Examples:

Is 89 closer to 80 or 100?

Is $\dfrac{7}{9}$ closer to $\dfrac{1}{2}$ or 1?

Is 35 + 45 bigger or smaller than 100?

Is $\dfrac{3}{4} + \dfrac{2}{3}$ bigger or smaller than 1?

If one examines these suggested pre-estimation problems, one realizes that the first two questions are number magnitude questions and the third and fourth questions may or may not draw from a student's ability to compute mentally.

The fourth suggestion is that students should explore and understand the effects of rounding. For example, students would be told that 50×30 is an estimate of 53×27 but then be asked if the exact answer is equal to, less than, or greater than the estimate. In one study, the pretest component revealed that half of the students believed that the answers were equalivant since they had gone three down and three up from 53 and 27 respectively. Results improved dramatically in the postinstruction component of the study where 75% of the students answered the question correctly providing the appropriate rational (Markovits & Sowder, 1994). One of these authors takes this further to say students should be encouraged to explore what happens when both numbers are rounded down, both are rounded up, and when one is rounded up while the other down (Sowder, 1995). It is interesting to note that this recommendation from Markovts and Sowder seems to conflict with the recommendations in the NCTM document. As previously mentioned, the authors of the document cautioned against having students become fixated on the proximity of their estimate to the exact answer.

The fifth suggestion, which is closely related to the fourth, is that students should be able to distinguish between the absolute and relative errors for an estimation. Consider the following estimation questions and the responses supplied.

Estimate: 34×86	Estimate: 496×86
Response: $30 \times 86 = 2580$	Response: $500 \times 86 = 43\ 000$

The exact answer to the first question is 2924, and since the response is 2580, the difference between the estimation and the exact answer is 344. The exact answer to the second question is 42656, and since the response is 43 000, the difference is 344. Students were then asked which of the estimations was better or were they equally as good. Prior to intervention most students believed that both estimations were equally as good because the same absolute error (344) was obtained in each case. They failed to consider the relative error.

Relative Error for Estimation Question 1

$$\frac{344}{2924} = 0.118$$

Relative Error for Estimation Question 2

$$\frac{344}{42646} = 0.008$$

The relative error for the second estimation problem is much lower, therefore the estimation is better even though the absolute errors are the same. It should be mentioned that students were not expected to calculate the relative errors but rather recognize that 344 out of 42646 is much smaller than 344 out of 2924 (Markovits & Sowder, 1994)

Since many students have difficulties estimating using rational numbers (Lindquist as cited in Sowder, 1995), the sixth suggestion is that more time is spent on rational numbers. There has only been limited work in the area of understanding estimation involving fractions, decimals, and percents (Hanson & Hogan, 2000), one researcher, however, recommends that students use benchmark values when attempting these types of problems (Sowder, 1995). For example, $\frac{5}{19}$ is very close to $\frac{5}{20}$ which can be expressed as the benchmark, $\frac{1}{4}$. Benchmark rational numbers are those numbers that students tend to have a more intuitive understanding of. They include fractions, decimals and percentages. Some benchmark values which are typically used are $\frac{1}{10}$, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{2}$, 0.1, 0.4, 0.75, 25%, 50%, 80%, and 100%. Students should be encouraged to use benchmark percents or decimals even if the original question is only stated using fractions.

Examples:

Estimate $\frac{8}{9}+\frac{22}{23}$	Estimate $12.2\times\frac{4}{11}$	Estimate $\frac{4}{9}+\frac{13}{23}-\frac{1}{11}$
Response	Response	Response
$1+1$	$12\times\frac{1}{3}$	$0.5+0.5-0.1$
2	4	0.9

Since many students believe that "division makes things smaller", the seventh suggestion is that students spend time investigating this misconception. For example, students could be asked to complete a series of questions similar to the ones listed below. These types of questions could be addressed when dealing with estimation problems or while dealing with mental computation problems.

Questions.

1. (a) $8\div 4=$
   (b) $8\div 2=$
   (c) $8\div 1=$
   (d) $8\div\frac{1}{2}=$
   (e) $8\div\frac{1}{4}=$

   Is there a pattern here? Explain.

2. Examine the following questions and then answer the multiple choice question below.

   $120 \div 4 \qquad \frac{1}{2} \div 2 \qquad 10 \div 9$

   If you take a number n and divide it by a number greater than one, then the resulting quotient is ____________________ the original number n.

   (a) greater than (b) equal to (c) less than

3. Examine the following questions and then answer the multiple choice question below.

   $120 \div \frac{1}{4} \qquad \frac{1}{2} \div \frac{1}{4} \qquad 10 \div \frac{1}{9}$

   If you take a number n and divide it by a number less than one, then the resulting quotient is ____________________ the original number n.

   (a) greater than (b) equal to (c) less than

Using language that prompts conceptual understanding is also critical to rectifying this misconception. For example, when given $7 \div \frac{1}{3}$, the teacher might ask the students, "How many $\frac{1}{3}$'s are in 7?" rather than, "What is 7 divided by $\frac{1}{3}$?" By rephrasing the question, the teacher is forcing the student to think that there are three $\frac{1}{3}$'s in 1, therefore there must be twenty-one $\frac{1}{3}$'s in 7.

In this section of the paper, we have identified problems associated with computational estimation and suggestions to address many of these problems. If these suggestions are implemented in classroom practices, how will teachers recognize when students are proficient computational estimators? One group of researchers recommends that teachers look for the following characteristics that are indicative of good estimators.

(1) Good estimators can quickly and accurately recall basic mathematical facts for all operations.
(2) They have the ability to change numerical values in the problem to more manageable forms.
(3) They are quick, efficient and accurate when using mental computation.
(4) They don't perceive themselves as "being wrong" when using estimation opposed to exact computations.

(5) They are prepared to adjust their initial estimate to compensate for numerical variation throughout the problem.

(6) Although they may not use the specific terms, they use the associative, distributive and commutative properties appropriately.

(7) They possess a variety of strategies for addressing estimation problems.

(8) They are confident in their own ability to obtain an appropriate estimate.

(Reys, Rybolt, Bestgen & Wyatt, 1982)

In closing, the research suggests that computational estimation should be pervasive in mathematics curriculum however, students must first be exposed to number sense activities, which involve number magnitude and mental computation. The estimation activities should encourage students to explore, use and discuss various strategies rather than relying on a new set of algorithms for estimation. Students should learn to access strategies, which are best suited for their needs and abilities. They also learn that there are a range of acceptable answers, not governed by the teacher or the exact answer, but rather by validity of the strategy used.

Judging Reasonableness of Results

Judging the reasonableness of a result means that students should examine the answer they have obtained with or without technology and determine whether the answer is appropriate given the question and the context. This means that students should reflect upon the answer obtained and also the process by which it was calculated.

Students often apply a learned algorithm without considering the reasonableness of the answer. Consider the three questions and solutions provided below.

Examples:

$50 - 17 = 47$ $\quad\quad$ $\frac{2}{3} + \frac{3}{4} = \frac{5}{7}$ $\quad\quad$ $37 + 48 = 715$

In the first example, the student has failed to consider that 50 and 47 only differ by 3, not 17. In the second example, the student has not considered that both of the original fractions are greater than $\frac{1}{2}$, therefore the resulting sum must be greater than 1. In the case of the third example, would the student have made the same mistake if the had to purchase one item for 37¢ and another for 48¢? One would anticipate that they would realize that a total cost of $7.15 was unreasonable (McIntosh, Reys, Reys & Hope, 1997). Consider another question and its corresponding responses.

Example: When you multiply 13.26 and 3.5, the answer is 4641, but the decimal point is missing. Place the decimal point in the appropriate position.

Two Responses:

Student A: 4.641 $\quad\quad$ Student B: 46.41

The first student hasn't considered whether their answer is reasonable. It appears the student applied the algorithm and therefore "moved the decimal point three spaces to the left". One might conclude that the second student has displayed number sense because he/she estimated that the answer should be around 52, the product of 13 and 4. This could only be determined if the second student was asked to justify their answer. In the examples provided so far, estimation was used to judge the reasonableness of the answer. In other cases, students must consider more than just estimation. Examine the two questions below and their corresponding responses.

Example: A tray can hold 8 bowls of soup. How many trays are required for 60 bowls of soup?

Two Responses: Student A: 7.5 trays Student B: 8 trays

Example: The height of a 10-year old boy is 140 cm. How tall will he be at 20 years of age?

Two Responses: Student A: 280 cm Student B: Between 190 cm and 210 cm

In both examples, the first student did not consider whether the answer was reasonable given the context of the question (Markovits, 1989, NCTM, 1989). If the student had considered this, they might modify their answer and/or the strategy employed.

In today's society where technology has inundated our lives, the ability to judge the reasonableness of results is important. If individuals are using technology and have entered a value incorrectly, it is hoped that the error might be identified if the individual examines the result and considers whether it is reasonable based on the question. If judging reasonableness is important, how do teachers encourage students to foster this critical component for the development of number sense? Judging reasonableness, like the other critical components to developing number sense, should be pervasive rather than an isolated topic. Discussions regarding answers and strategies are required because they invariably lead back to the topic of reasonableness. Students should be exposed to questions similar to the examples shown in this section that force them to consider the reasonableness of answers. Teachers should also model this behavior by continually sharing with students the methods they use to check the reasonableness their answers (McIntosh, Reys & Reys, 1997).

Assessment

Much research has been done regarding changing instructional practices for number sense however, very little work has been done designing assessment items for the classroom. This probably reflects the belief that assessment regarding number sense extends well beyond the traditional evaluation techniques of the past (Howden, 1989). Understanding the type and level of number sense possessed by an individual requires students to provide justification for their answers either in written or oral form. Traditional assessment techniques have generally required the use of algorithms as justification for an answer.

Research suggests that one must observe students solving problems, inquire about their line of reasoning, and then judge to what extent the children are reasoning effectively about number (Resnick, 1989, Howden, 1989). Many researchers have implemented such methods when attempting to study number sense (Forrester & Pike, 1998, Heirdsfield, 2000, Hope, 1987, Howden, 1989, Mack, 1990, Markovits & Sowder, 1994, Sowder, 1995, Reys & Yang, 1998, Weber, 1996). Teachers should listen and observe students individually and in group settings. They should also require students to provide complete explanations of their work and solutions orally or in writing, rather than merely producing an answer. The teacher should also be prepared to ask probing questions of their students. The following dialogue between a class and their teacher, is an example of this. The questions challenge students to explain their thinking, consider alternate approaches, change the context, and think beyond the supplied question.

Teacher:	A fruit punch recipe calls for $2\frac{1}{4}$ litres of water. You need to quadruple the batch. How much water do you need?
Student 1:	I need nine litres.
Teacher:	Is that an estimate or an exact answer?
Student 1:	It's an exact answer.
Teacher:	How did you get that answer?
Student 1:	Four times two is eight and four quarters give one. Eight plus one is nine.
Teacher:	Excellent, can you or anyone else think of another way to solve the problem?
Student 2:	If I double $2\frac{1}{4}$, you get $4\frac{1}{2}$. If you double $4\frac{1}{2}$, you get 9.
Teacher:	Correct. Why did you take that approach?

Student 2: I find it easy to double numbers and if you double twice, it's the same as quadrupling once.

Teacher: Very good. I want to pose this question to you. If the original recipe had to be six times larger, would you double $2\frac{1}{4}$ three times?

Student 2: No

Teacher: Why?

Student 2: If you double it three times, it makes it eight times larger.

Teacher: Excellent. Did anyone handle the question a different way?

Student 3: I did it a different way. The number $2\frac{1}{4}$ can be expressed as nine quarters. If you quadruple nine quarters, you get thirty-six quarters which is equal to nine.

Teacher: Well done. Have any of you considered changing the question slightly so that you're dealing with the same numbers but in a different context or situation?

Student 4: Yes, I changed it so that I was dealing with money. I wanted to know how much I would have if I quadrupled $2.25. If I quadruple $2, I get $8. If I now have four quarters instead of one quarter, I get $1. The total is then $9. Going back to the original question that gives me an answer of 9 litres.

Teacher: Perfect. What have we learned from this?

Student 5: There is more than one way to solve this problem.

By focusing on the strategies employed, rather than the answers obtained, the teacher can start to assess the level of number sense possessed by students.

The fact that number sense varies from student to student, from grade to grade, and from one task to another, one can conclude that number sense is difficult to assess. However, the National Council of Teachers of Mathematics (1989) has proposed the common characteristics of individuals who possess good number sense. These characteristics should be considered when attempting to assess number sense.

"Children with good number sense:

(1) have a well understood number meaning,

(2) have developed multiple relationships among numbers,

(3) recognize the relative magnitudes of numbers,

(4) know the effects of operating on numbers, and

(5) develop referents for measures of common objects and situations in their environments."

There is another factor that should be considered when attempting to assess number sense; that is the role of experience. If a student has done seven questions using the same strategy, when the child completes an eighth question using the same strategy, is he/she exhibiting number sense? (Markovits, 1989) If number sense is characterized by flexibility and nonstandard approaches, how can that be assessed if students are drilled repeatedly with the same types of questions? This implies that teachers and curriculum developers should be cautious when developing new materials and assessment items to insure that students experience number and operations opposed to memorizing a series of number sense algorithms.

Number Sense and Learners with Mathematical Disabilities

A learning disability is a lifelong neurologically based disorder that does not usually affect the intelligence of an individual, but rather the perception and processing of information. Learners may have difficulties in the acquisition and use of listening, speaking, reading, writing, reasoning, or mathematical abilities, or social skills (ICLD as cited in NCSALL, 2000). Approximately 15% of the population in the United States, and 10% to 15% of the Canadian population has some form of learning disability. Of this group, approximately 60% possess significant disabilities in mathematics (Light & DeFries as cited in Gersten & Chard, 2001).

Research advocates the use of number sense activities as a mechanism for increasing a learner's acquisition of mathematical concepts, especially for those learners dealing with a learning disability (Gersten and Chard, 2001, Griffin et al. as cited in Gersten and Chard, 2001). Effective mathematics instruction for LD learners occurs when number sense activities are integrated into daily practices, curriculum and assessment practices. Some researchers content that number sense is as important to mathematics learning as phonemic awareness has been in the field of reading (Gersten and Chard, 2001). They believe that greater progress in mathematics education for LD learners will be made when a wave of research and development in number sense parallels the research and development in instructional strategies related to the concept of phonemic awareness.

Two researchers identify the important relationship between number sense and automaticity as it applies to LD learners.

> We submit that simultaneously integrating number sense activities with increased number fact automaticity rather than teaching these skills sequentially- advocated by earlier special education mathematics researchers such as Pellegrino and Goldman (1987)- appears to be important for both reduction of difficulties in math for the general population and for instruction of students with learning disabilities. It is also likely that some students who are drilled on number facts and then taught various algorithms for computations may never develop much number sense. (p. 5, Gersten and Chard, 2001)

Although Gersten and Chard disagreed with Pellegrino and Goldman’s delivery model, they did agree with their finding that improving automaticity would allow the working memory to engage in higher order thinking.

> Pellegrino and Goldman (1987) concluded that the focus of mathematics remediation for students with learning disabilities should involve extended practice on math facts for which the student still relied on counting procedures. They argued that extended practice would lead to "development of a degree of automaticity that affords them the attentional and resource opportunities to engage in metacognitive activities… being able to allocate more attention to higher-order aspects of the task or to restructuring of performance patterns" (Pellegrino & Goldman, 1987, p. 146). Using conceptions of cognitive processes prevalent at the time, they argued that basic math facts must become declarative knowledge so that the students can devote energies to higher-order thinking. (p. 6, Gersten and Chard, 2001)

Further research demonstrated that drill-and-practice computer-assisted instruction for approximately 10 minutes per day did increase automaticity for most, but not all LD students (Hasselbring et al, as cited in Gersten and Chard, 2001). This finding would support the use of online applets like the Math Magician Games. Although automaticity in conjunction with number sense is important to improving mathematical understanding, is not necessarily required. “Even if students are not automatic with basic facts, they still should be engaged in activities that promote the development of number sense and mathematical reasoning. “ (p. 12, Gersten and Chard, 2001)

Conclusions

Number sense is grounded in a sound understanding of number and operations and is exhibited by students when they operate flexibly with number utilizing numerous standard and nonstandard approaches. Number sense develops gradually and is characterized by both its highly intuitive and personal nature. Number sense involves judgment and interpretation and is dependent upon a complex interaction among an individuals' knowledge, an individual's skills, the nature of the problem, and the expected performance on a particular problem.

One of the critical issues regarding the development of number sense is student and teacher conceptions of mathematics. If these two groups perceive mathematics as a dynamic, creative, and sense-making discipline, they are more likely to discover, accept, advocate for, and implement nonstandard strategies necessary for the development of number sense. Sometimes students and/or teachers believe that mathematics is static, devoid of questioning, and driven by the use of algorithms. When this occurs, curriculum, resources, and activities, which are inundated with novel problems, should be introduced to challenge this conception. The classroom practices and resources that encourage exploration and discussion also allow students to appreciate and generate unique solutions. This creates opportunities for students to make connections within and between their formal and informal knowledge of mathematics. Greater number sense may be established when students are asked to take risks and trust their own knowledge. They must be prepared to make mistakes and view them as opportunities to learn. By doing this, nonstandard approaches, short-cuts, and multiple valid solutions should be celebrated by the teacher and class. The strategies implemented and solutions obtained should be discussed in a supporting environment where the teacher is prepared to ask probing questions and requires students to provide complete explanations.

When attempting to judge the nature of number sense possessed by an individual, one should observe that individual completing items from the four major components of number sense; number magnitude, mental computation, estimation, and reasonableness of answers. A brief summary of the findings for each of these areas is presented below.

Understanding number magnitude means that individuals should be able to compare numbers such that they can order the numbers, recognize which of two numbers is closer to a third, and to identify numbers between two given numbers. Most students are able to judge number magnitude with whole numbers but research indicates that there are significant problems at all grade levels with decimals and fractions. Some students have difficulties even initiating these types of questions and/or have difficulties recognizing the relationships between decimals and fractions. Several recommendations were offered to improve students conceptual understanding of proportional quantities.

These recommendations suggest that students:

(1) learn and work with decimals, fractions, and percents simultaneously,

(2) use more intuitive approaches to understand the relationship,

(3) use benchmark values to first develop an understanding of the relationship,

(4) use pictorial representations and manipulatives that were traditionally only used when working with fractions,

(5) use the appropriate mathematical language,

(6) are exposed to a variety of novel questions concerned with proportional quantities, and

(7) are asked to discuss the different standard and nonstandard approaches employed to solve such questions.

Mental computation is the process of calculating the exact numerical answer without the aid of any external calculating or recording device. Many students feel that mental computation is important however, they surprisingly have a limited range of strategies and many of the strategies reflect algorithmic approaches learned in school. This implies that many students become fixated on algorithms. To address this issue, researchers recommend that some mental computation activities should proceed the teaching of algorithms. If this is done then students are permitted to develop more intuitive insights that draw from their informal knowledge and allow students the opportunity to make new connections in their conceptual networks. This can be accomplished by allowing students to explore, discover and discuss new strategies best suited for their needs and abilities. Although mental computation can be mentally taxing, students often develop nonstandard strategies that ultimately reduce the burden on the short-term memory.

Computational estimation refers to one’s ability to estimate answers to numerical computations. Computational estimation is a difficult task for many young students therefore it is recommended that computational estimation should not be introduced until the students are developmentally ready; possibly as late as grade 7. It is also recommended that number magnitude and mental computation should proceed computational estimation. The researchers contend that many of the strategies, discovered in those areas, are required to successfully complete computational estimation. They also caution teachers from teaching the explicit use of rounding rules as the only means of obtaining a reasonable estimate. Students will ultimately view the rounding rules as the algorithm for estimation and, in turn, believe that there is only one acceptable estimate. Classroom practices and activities that focus on the numerous strategies that can be employed on a problem, better serve the needs of all students and demonstrate to the students that there is a wide range of acceptable answers. The researchers also state that this range of acceptable answers is governed by the validity of the strategy used versus the proximity of the estimate to the exact answer.

Judging the reasonableness of a result means that students should examine the answer they have obtained with or without technology and determine whether the answer is appropriate given the question and the context. The statement, “Does that seem reasonable?” should be used extensively in the classroom regardless if the activities are specifically designed to foster number sense. Opportunities to reflect upon strategies, answers and the context of the problem are important in a fast-paced world that is inundated with technology.

In closing, research has demonstrated that if the appropriate teaching practices and resources are implemented, we can change the way students process and think about numbers and operations. Students should be asked to explore meaningful and purposeful problems where the students should rely on their formal and informal knowledge to generate their own strategies. This fosters number sense. In addition to this, classrooms that encourage discussion, ask students to justify their positions, are tolerant of mistakes, and support the use of standard and nonstandard approaches are critical in the development of number sense.

Expressing a Number Different Ways

Example: Express the number 6 many different ways.

Answer:	half a dozen	$5+1$	$7-1$	$4+2$	$8-2$
	2 sets of 3	3 sets of 2	2×3	$3\times 3-3$	$2\times 5-4$
	$12\div 2$	$18\div 3$	$20\div 4+1$	$22\div 2-5$	$1+2+3$
	$10-8+4$	4^2-10	1^2+5	3^2-3	2^2+3-1

Questions:
Express each of the numbers many different ways.
(a) 12

(b) 8

(c) 18

(d) 25

Sum and Product Squares (Part 1)

1. You will need a red pen or pencil, and a blue pen or pencil to do this activity sheet. You will notice that for each of the 4 by 4 charts (sum and product squares) below, there is a number supplied above the chart. This number will be called the target number. Complete two tasks for each of the sum and product squares.

   Task 1: Using a blue pen or pencil, circle the two adjacent numbers whose sum is the target number.

   Task 2: Using a red pen or pencil, circle the two adjacent numbers whose product is the target number.

(a) Target Number: 6

7	3	2	4
0	3	8	12
6	4	0.5	1
3	1	5	2

(b) Target Number: 12

3	10	1	12
9	4	11	6
0.5	12	5	6
0	24	2	7

(c) Target Number: 20

16	4	9	2
5	10	10	18
7	3	1	15
40	0.5	5	20

(d) Target Number: 15

14.5	0.5	10	1
3	30	0	15
5	10	4	5
14	3	12	11

2. Create your own sum and product square.

Sum and Product Squares (Part 2)

1. You will need a red pen or pencil, and a blue pen or pencil to do this activity sheet. You will notice that for each of the 4 by 4 charts (sum and product squares) below, there is a number supplied above the chart. This number will be called the target number. Complete two tasks for each of the sum and product squares.

   Task 1: Using a blue pen or pencil, circle the two adjacent numbers whose sum is the target number.

   Task 2: Using a red pen or pencil, circle the two adjacent numbers whose product is the target number.

(a) Target Number: -10

2	0.5	7	10
8	-5	-20	6
-3	-7	-15	10
-2	5	4	-1

(b) Target Number: 4

5	3	0.5	8
1	6	-4	2
-2	6	5	2
-2	7	-1	-4

(c) Target Number: -100

-4	10	-85	-15
25	-150	-10	50
50	-90	-5	60
40	-2	-120	20

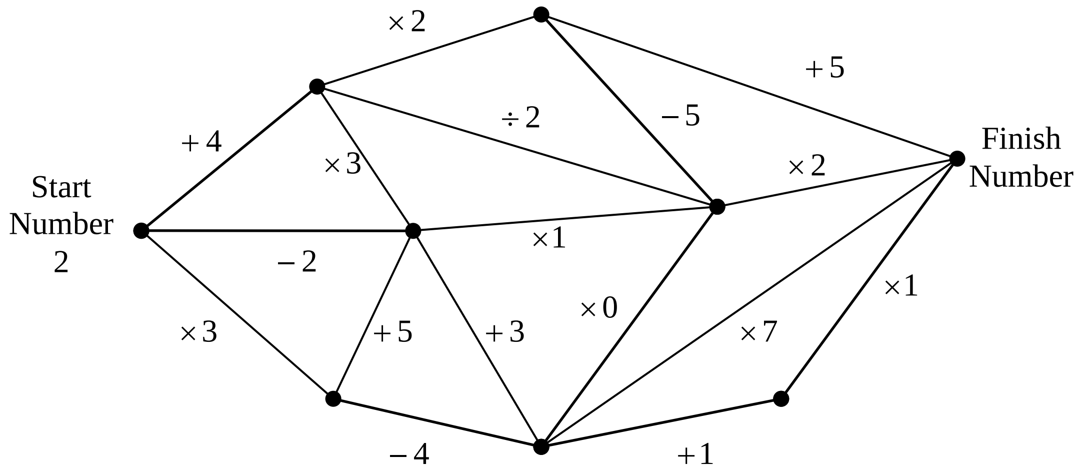
(d) Target Number: 200

180	20	-2	30
70	-100	10	201
-40	230	-30	-1
205	-5	-200	10

2. Create your own sum and product square that uses integers.

Picking a Route

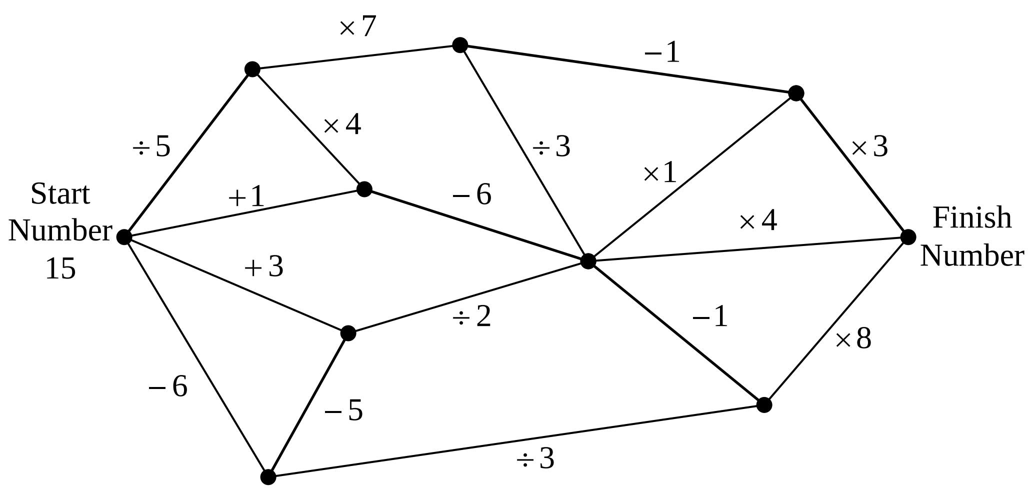
An obstacle course is provided below. Your mission is to begin at the Start Number, move from left to right along paths of your choosing, complete the necessary operations, and determine your Finish Number.



Complete the obstacle course four times choosing different routes. Trace the routes on the diagram.

Route #1 Finish Number: ______ Route #2 Finish Number: ______
Route #3 Finish Number: ______ Route #4 Finish Number: ______

Do this again for the new obstacle course provided below.



Route #1 Finish Number: ______ Route #2 Finish Number: ______
Route #3 Finish Number: ______ Route #4 Finish Number: ______

Blazing a Trail (Part 1)

1. The Sum Trails
   Using only the addition operation, find the trail of numbers that add to the final sum in the bottom right hand corner of the puzzle. You must start from the upper left box and you can only move horizontally or vertically from box to box.

(a)

2	4	0
3	5	6
8	1	7
		23

(b)

3	0	2
6	7	5
1	8	4
		19

(c)

1	2	9
3	8	5
6	4	7
		21

(d)

5	4	9
1	6	7
3	8	2
		32

2. The Product Trails
   Using only the multiplication operation, find the trail of numbers that add to the final product in the bottom right hand corner of the puzzle. You must start from the upper left box and you can only move horizontally or vertically from box to box.

(a)

4	0	1
1	5	2
6	3	1
		60

(b)

2	1	2
6	3	5
0	1	4
		120

(c)

3	1	5
2	0	6
4	1	2
		48

(d)

1	3	0
6	1	2
5	4	3
		36

3. Create your own sum trail and product trail.

(a) Sum Trail

(b) Product Trail

Blazing a Trail (Part 2)

1. The Sum Trails
   Using only the addition operation, find the trail of numbers that add to the final sum in the bottom right hand corner of the puzzle. You must start from the upper left box and you can only move horizontally or vertically from box to box.

(a)

-2	1	-2
0	3	-4
6	5	-3
		6

(b)

4	1	-1
2	-3	5
0	-1	-2
		5

(c)

0.2	0.5	3.1
0	0.3	0.4
2	1.2	1
		4.8

(d)

0.1	0	-0.4
1	0.3	-0.3
-0.5	0.1	0.2
		-0.4

2. The Product Trails
   Using only the multiplication operation, find the trail of numbers that add to the final product in the bottom right hand corner of the puzzle. You must start from the upper left box and you can only move horizontally or vertically from box to box.

(a)

4	-1	-3
1	2	-1
3	0	2
		16

(b)

-2	2	5
-1	1	0
-3	-4	3
		48

(c)

-3	4	-1
5	1	2
0	-2	-2
		-60

(d)

6	-0.5	3
1	0	-1
-2	5	3
		27

3. Create your own sum trail and product trail using decimals and/or integers..

(a) Sum Trail

(b) Product Trail

Bull's Eye

This is a game for two or more players where each player tries to reach a specified number (the target number) with the least number of rolls of a die. The players will decide upon a target number. The target number should be a whole number between 40 and 120. Decide which player will go first. Each player rolls the die separately. The first roll of the die gives the player the number that he/she will start with. It will also be called the new number for this first roll. The next time the player gets to roll again, the number that is rolled will be added, subtracted, multiplied or divided from the new number in the last step. This new sum, difference, product or quotient will be recorded in the last row for each step. This process continues until someone reaches the target number. If no one reaches the target number after fourteen rolls of the die, the winner will be the closest one to the target number after the fourteenth roll. Consider the sample game below. Player 1, Monique, rolled a 2 on the first roll. She rolled a 5 on the second roll. Since she was trying to reach the target number of 41, she decided to multiply the 2 and 5, giving a product of 10. The third number rolled was a 3. She decided to multiply it by the 10 to get the 30.

Sample Game

Target Number: **41**

Player 1 : Monique

Number Rolled	2	5	3	1	6	5	3	6	2					
Operation Used	NA	×	×	+	+	+	-	+	-					
New Number	2	10	30	31	37	42	39	45	43					

Player 2 : Barbara

Number Rolled	6	3	2	2	5	3	4	1	2					
Operation Used	NA	×	+	×	+	-	-	+	+					
New Number	6	18	20	40	45	42	38	39	41					

The winner is Barbara.

Play the game at least five times, recording the results in the tables below.

Game 1

Target Number:________

Number Rolled														
Operation Used	NA													
New Number														

The winner is ___________________.

Game 2

Target Number:________

Number Rolled														
Operation Used	NA													
New Number														

The winner is ___________________.

Game 3

Target Number:________

Number Rolled														
Operation Used	NA													
New Number														

The winner is ___________________.

Game 4

Target Number:________

Number Rolled														
Operation Used	NA													
New Number														

The winner is ___________________.

Game 5

Target Number:________

Number Rolled														
Operation Used	NA													
New Number														

The winner is ___________________.

If you want to make the game more challenging, try rolling two dice. If you do this, choose larger target numbers.

The Fantastic Four Card Game

Take a deck of playing cards and remove all the face cards (jacks, queens and kings). Shuffle the deck. Deal yourself five cards. Write down the numbers for the five cards. Your mission is to use the numbers from the first four cards to make equations that equal the number on the fifth card. You can only use each number once unless more than one card was drawn with the same value. You don't have to use all four numbers. You have five minutes to create as many equations as you can.

Example: Your first four cards have the numbers 2, 3, 7 and 9. The fifth card is a 10.

Your Answers: $3+7=10$ $\quad (3-2)+9=10$

$\frac{9}{3}+7=10$ $\quad (2^3-7)+9=10$

Questions:

1. Your first four cards have the numbers 3, 2, 4 and 10. The fifth card is a 6. Generate as many equations as possible.

2. Your first four cards have the numbers 3, 4, 10 and 8. The fifth card is a 6. Generate as many equations as possible.

3. Use a deck of playing cards with the face cards removed. Deal yourself five cards. Record these numbers and make as many equations as possible. Do this activity five times.

(a)

1st	2nd	3rd	4th

5th

(b)

1st	2nd	3rd	4th

5th

(c)

1st	2nd	3rd	4th

5th

(d)

1st	2nd	3rd	4th

5th

(e)

1st	2nd	3rd	4th

5th

If you want to do more of these, you may want to get on the internet and Google *Playing Fantastic Four with the Computer*. The website will generate the cards and allow you to check your answers.

Sequences (Part 1)

1. Determine the missing terms in the sequence.

(a) 3, 5, 7, 9, _____, _____, _____

(b) 11, 14, 17, 20, _____, _____, _____

(c) 102, 98, 94, 90, _____, _____, _____

(d) 212, 209, 206, 203, _____, _____, _____

(e) 4, _____, 10, _____, 16, _____, 22

(f) _____, 13, _____, 23, _____, 33

(g) 72, _____, 66, _____, 60, _____, 54

(h) _____, 72, _____, 64, _____, 56

(i) 0.5, 1, 2, 4, _____, _____, _____

(j) 800, 400, 200, _____, _____, _____

(k) 5, 10, 20, 40, _____, _____, _____

(l) 10, 30, 90, 270, _____, _____, _____

(m) 3, 6, 12, 24, _____, _____, _____

(n) _____, 14, 28, 56, _____, _____

(o) _____, 48, 24, 12, _____, _____

(p) 5, _____, 45, 135, _____, 1215, _____

(q) _____, 120, 145, 170, _____, _____

(r) 143, _____, 149, _____, 155, 158, _____

2. In question 1, all of the sequences either had a common difference or common ratio between the successive terms. In this question, the pattern is a little harder to identify. You may wish to work with a partner when attempting to find the missing terms.

(a) 1, 4, 9, 16, 25, _____, _____, _____

(b) 1, 8, 27, 64, 125, _____, _____, _____

(c) 1, $\sqrt{2}$, $\sqrt{3}$, 2, _____, _____, _____

(d) 1, 2, 3, 5, 8, 13, _____, _____, _____

(e) 0, 3, 8, 15, 24, _____, _____, _____

(f) 3, 7, 12, 19, 28, _____, _____, _____

3. Create the sequence where the first term is 7 and there is a common difference of 3 between successive terms.

4. Create the sequence where the first term is 4 and there is a common ratio of 3 between successive terms.

5. Given the following situation, create the appropriate sequence. You initially have $300 in your bank account and withdraw $40 per day for six days.

6. Given the following situation, create the appropriate sequence. The bacteria population in the Petri dish started at 30 bacteria per square centimeter. The population doubles every hour for six hours.

Sequences (Part 2)

1. Determine the missing terms in the sequence.

(a) 0.6, 1, 1.4, 1.8, ______, ______, ______

(b) 4.4, 4.1, 3.8, 3.5, ______, ______, ______

(c) ______, 5, 2, -1, ______, ______, -10

(d) ______, -5, -3, ______, 1, 3, ______, 7

(e) -1, -0,8, -0.6, ______, ______, 0, ______

(f) ______, 1.2, 0.7, 0.2, ______, ______, -1.3

(g) $\frac{1}{2}$, $\frac{3}{4}$, 1, ______, $1\frac{1}{2}$, ______, ______

(h) 1, $\frac{7}{8}$, $\frac{3}{4}$, ______, $\frac{1}{2}$, ______, ______

(i) 100, 10, 1, ______, 0.01, ______, ______

(j) 0.6, 1.2, 2.4, ______, ______, 19.2, ______

(k) $\frac{5}{2}$, $\frac{5}{6}$, $\frac{5}{18}$, ______, ______, $\frac{5}{486}$, ______

(l) $\frac{5}{7}$, $\frac{15}{14}$, $\frac{45}{28}$, ______, ______, $\frac{1215}{224}$, ______

(m) 100, 150, 225, 337.5, ______, ______

(n) ______, $\frac{2}{9}$, $\frac{4}{27}$, $\frac{8}{81}$, ______, $\frac{32}{729}$, ______

(o) -1, 2, -4, 8, -16, ______, ______, ______

(p) ______, 3.75, 3.8, 3.85, ______, 3.95, ______

2. Create the sequence whose first term is 9 and its common difference between the successive terms is -0.4.

3. Create the sequence whose first term is 4.5 and whose common ratio between the successive terms is 0.2.

4. Create your own sequence that has a common difference of 4.5 between the successive terms.

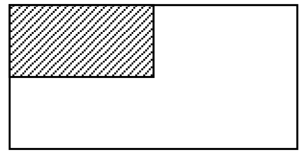
5. Create your own sequence that has a common ratio of 1.5 between the successive terms.

6. Create your own real-world situation that could be modeled by a sequence with a common difference.

7. Create your own real-world situation that could be modeled by a sequence with a common ratio.

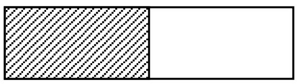
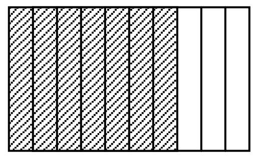
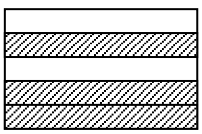
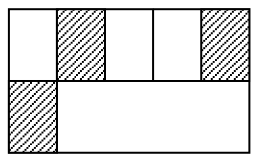
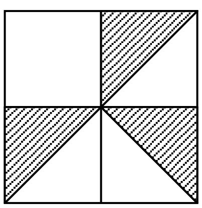
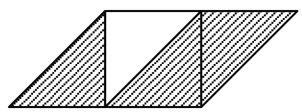
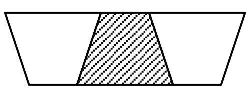
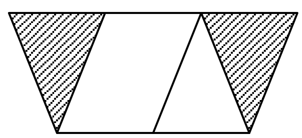
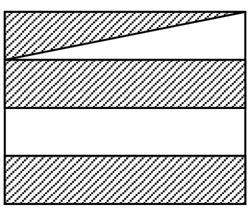
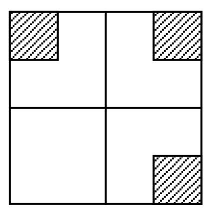
What Portion is Shaded?

Consider the area model on the right. If you were asked to determine what portion of the rectangle is shaded, you could express your answer three different ways. It could be expressed as a fraction, decimal or percent.



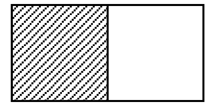
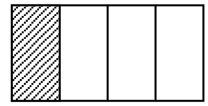
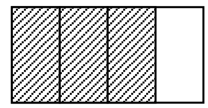
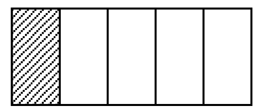
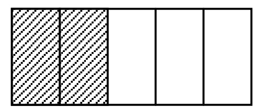
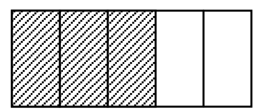
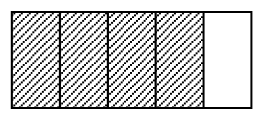
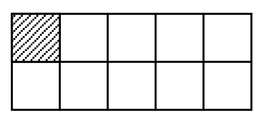
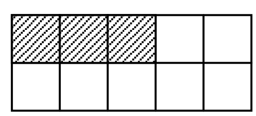
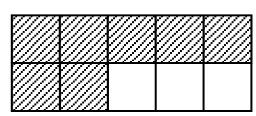
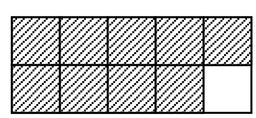
Fraction: $\frac{1}{4}$
Decimal: 0.25
Percent: 25%

For each quadrilateral, determine what portion is shaded. Express your answer three different ways.

	Fraction: Decimal: Percent:		Fraction: Decimal: Percent:
	Fraction: Decimal: Percent:		Fraction: Decimal: Percent:
	Fraction: Decimal: Percent:		Fraction: Decimal: Percent:
	Fraction: Decimal: Percent:		Fraction: Decimal: Percent:
	Fraction: Decimal: Percent:		Fraction: Decimal: Percent:

Fraction, Decimal and Percent Cards

Use these cards to create your own game in which the learner must match cards.

$\frac{1}{2}$	0.5	50%	
$\frac{1}{4}$	0.25	25%	
$\frac{3}{4}$	0.75	75%	
$\frac{1}{5}$	0.2	20%	
$\frac{2}{5}$	0.4	40%	
$\frac{3}{5}$	0.6	60%	
$\frac{4}{5}$	0.8	80%	
$\frac{1}{10}$	0.1	10%	
$\frac{3}{10}$	0.3	30%	
$\frac{7}{10}$	0.7	70%	
$\frac{9}{10}$	0.9	90%	

Proportional Reasoning Squares

1. Below you will find four proportional reasoning squares. These squares are comprised of decimals, fractions and percents. Your mission is to circle the two adjacent numbers that are equal to each other. Each proportional reasoning square has five matches. Find all five for each square.

(a)

$\frac{1}{4}$	$\frac{1}{5}$	0.1	$\frac{1}{10}$
25%	0.6	0.5	$\frac{3}{5}$
0.2	$\frac{1}{2}$	0.1	0.3
$\frac{1}{8}$	20%	0.7	30%

(b)

$\frac{4}{5}$	$\frac{2}{5}$	10%	0.7
$\frac{3}{10}$	0.3	0.4	70%
0.2	$\frac{3}{4}$	$\frac{1}{5}$	0.9
75%	30%	15%	0.15

(c)

$\frac{2}{5}$	0.5	0.75	$\frac{3}{4}$
0.1	40%	$\frac{3}{5}$	0.6
$\frac{7}{10}$	60%	0.7	15%
70%	$\frac{1}{5}$	0.15	$\frac{1}{2}$

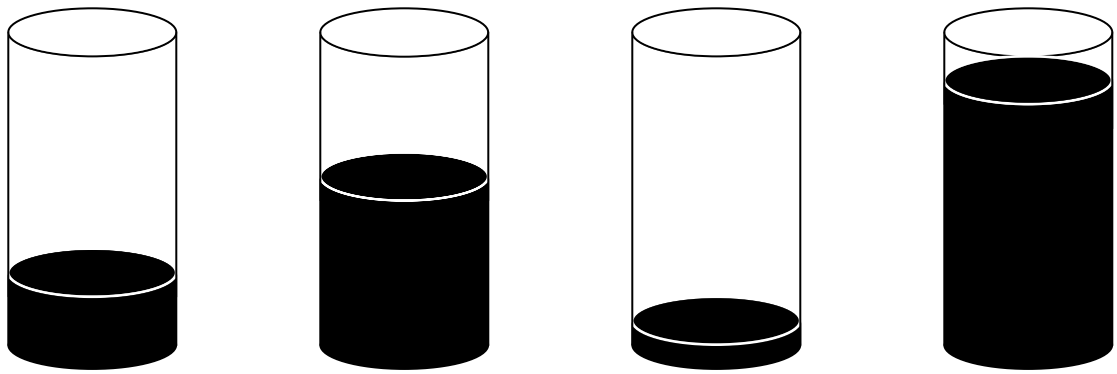
(d)

0.03	15%	$\frac{5}{5}$	0.6
3%	0.8	$\frac{9}{10}$	100%
0.01	$\frac{1}{100}$	90%	$\frac{1}{4}$
$\frac{7}{10}$	80%	$\frac{4}{5}$	0.5

2. Create your own proportional reasoning square.

Approximately How Full?

1. Four one litre cylindrical containers have been partially filled with water. Below the cylinders you will find a series of statements. Without using a ruler, match each statement with the appropriate container. We are only approximating. Each container will have three correct corresponding statements that describe approximately how full the container is.



Statements:

(a) Approximately 10% of the cylinder is filled.
(b) Approximately $\frac{5}{6}$ of the cylinder is filled.
(c) Approximately 0.55 litres of water is in the cylinder.
(d) Approximately $\frac{11}{20}$ of the cylinder is filled.
(e) Approximately 0.1 litres of water is in the cylinder.
(f) Approximately 20% of the cylinder is filled.
(g) Approximately $\frac{1}{4}$ of the cylinder is filled.
(h) Approximately 0.8 litres of water is in the cylinder.
(i) Approximately $\frac{1}{9}$ of the cylinder is filled.
(j) Approximately 0.45 litres of water is in the cylinder.
(k) Approximately 85% of the cylinder is filled.
(l) Approximately 30% of the cylinder is filled.

2. Create one statement for each container in question 1 that could approximate how full each is.

Finding Numbers Between Other Numbers

1. With each of these questions you are supplied with two numbers and are asked to find a number between them. You will be given four choices. Circle the correct answer.

		Choices			
(a)	Choose the fraction between $\frac{2}{7}$ and $\frac{4}{7}$.	$\frac{6}{7}$	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{5}{7}$
(b)	Choose the decimal between 0.7 and 0.9.	0.8	1.0	0.65	0.91
(c)	Choose the percentage between 39% and 43%.	44%	38%	46%	41%
(d)	Choose the fraction between $\frac{1}{2}$ and $\frac{4}{5}$.	$\frac{3}{8}$	$\frac{10}{9}$	$\frac{2}{3}$	$\frac{1}{10}$
(e)	Choose a decimal between 0.4 and 0.5	0.45	0.55	0.35	0.3
(f)	Choose a fraction between $\frac{1}{10}$ and $\frac{1}{3}$.	$\frac{2}{3}$	$\frac{4}{5}$	$\frac{8}{3}$	$\frac{1}{7}$
(g)	Choose a decimal between 60% and 70%.	0.75	0.8	0.065	0.68
(h)	Choose a decimal between $\frac{1}{5}$ and $\frac{2}{5}$.	0.8	0.1	0.6	0.3
(i)	Choose a percent between 0.6 and 0.82.	7%	5.8%	71%	0.7%
(j)	Choose a fraction between 0.1 and 0.4.	$\frac{4}{4}$	$\frac{8}{7}$	$\frac{3}{10}$	$\frac{5}{6}$
(k)	Choose a fraction between 45% and 60%.	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{1}{4}$
(l)	Choose a decimal between $\frac{1}{2}$ and $\frac{3}{4}$.	0.2	0.6	0.45	0.9
(m)	Choose a percentage between $\frac{2}{3}$ and $\frac{4}{5}$.	70%	90%	30%	50%

2. Find a number between the two numbers supplied. Below each space, it will state what type of number (fraction, decimal or percent) is desired. There is more than one correct answer for each of these questions.

(a)

0.6		0.8
	fraction	

(b)

$\frac{1}{5}$		$\frac{1}{3}$
	decimal	

(c)

$\frac{4}{5}$		$\frac{8}{9}$
	percent	

(d)

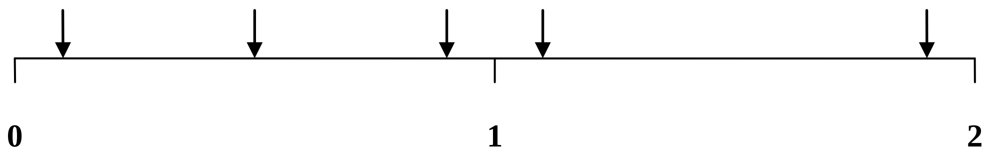
0.26		$\frac{3}{7}$
	fraction	

The Number Line (Part 1)

For each question, a list of numbers and a number line has been provided. Match each number with its approximate location on the number line.

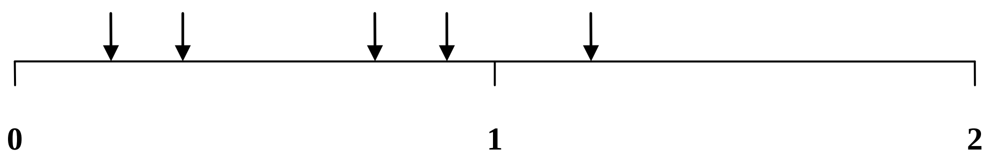
1.

0.5	1.09	0.9	1.9	0.09



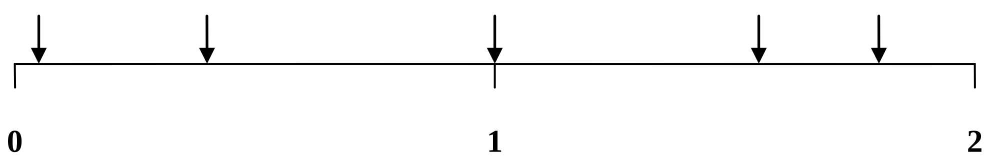
2.

$\frac{3}{4}$	$\frac{6}{5}$	$\frac{1}{5}$	$\frac{9}{10}$	$\frac{1}{3}$



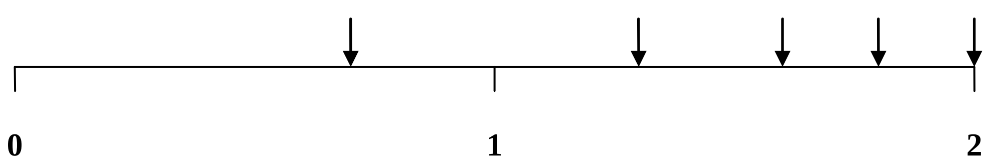
3.

$1\frac{1}{2}$	1.8	0.05	0.4	$\frac{7}{7}$



4.

1.6	$\frac{8}{4}$	0.7	$\frac{7}{6}$	$\frac{9}{5}$

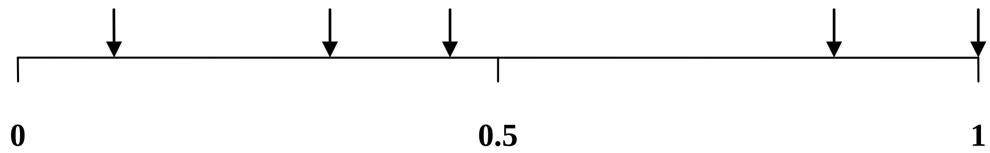


The Number Line (Part 2)

For each question, a list of numbers and a number line has been provided. Match each number with its <u>approximate</u> location on the number line. Note that the scales on the number lines change from question to question.

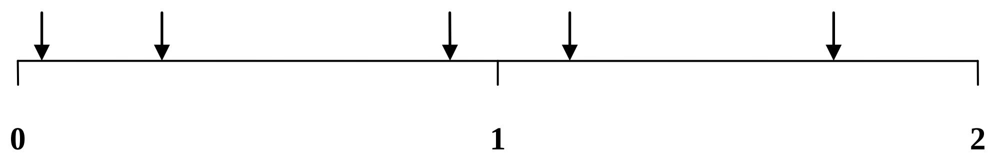
1.

$\frac{1}{3}$	1^2	0.09	$\frac{6}{7}$	0.45



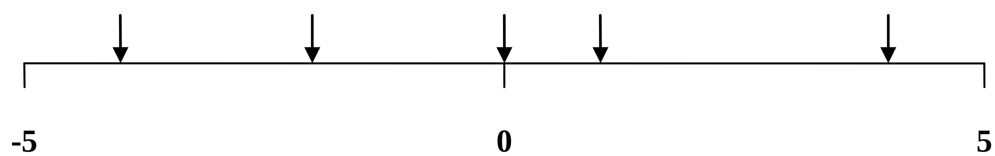
2.

0.9	0.05	$\frac{8}{7}$	1.7	$\frac{3}{10}$



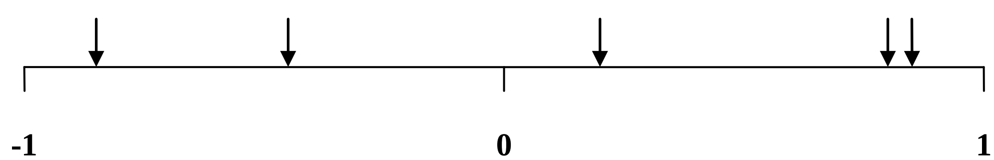
3.

$\frac{0}{4}$	2^2	-4.05	$\frac{8}{8}$	-1.99



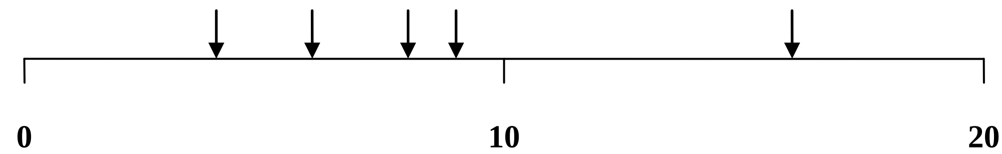
4.

$\frac{4}{5}$	-0.4	$\frac{1}{5}$	-0.8	$\frac{8}{9}$



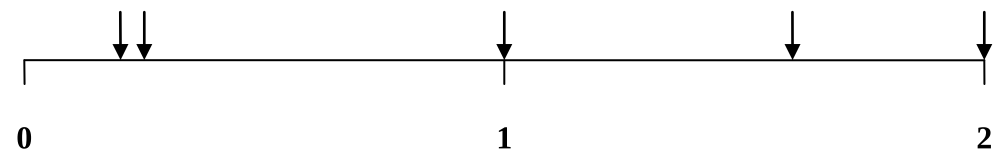
5.

3^2	$\sqrt{16}$	2^3	$\sqrt{36}$	2^4



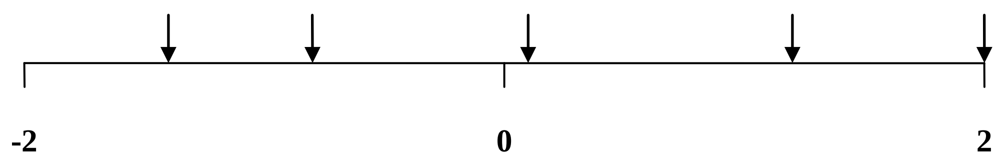
6.

$\sqrt{1}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{16}{8}$	1.58



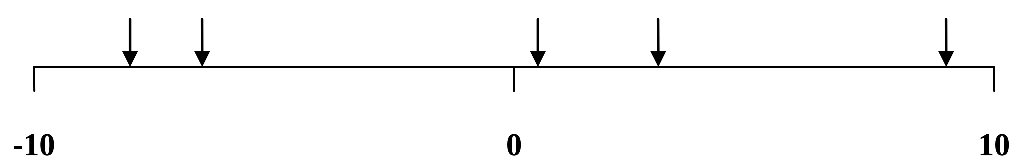
7.

$\sqrt{4}$	0.05	$-\frac{4}{5}$	$\frac{6}{5}$	-1.4



8.

$(-2)^3$	$\frac{27}{9}$	$\sqrt{81}$	-6.5	$\sqrt{\frac{1}{4}}$



Just the Answer (A)

Do each of these questions in your head and write down the final answer.

(a)	$20+30-10=$	(b)	$60+20-30=$
(c)	$40+50-20=$	(d)	$50+30-40=$
(e)	$20+40-50=$	(f)	$70-30+40=$
(g)	$90-20+10=$	(h)	$30+60-40+10=$
(i)	$40+20-30+20=$	(j)	$80-20+30-10=$

Just the Answer (B)

Do each of these questions in your head and write down the final answer.

(a)	$200+500-100=$	(b)	$300+400-200=$
(c)	$600+300-500=$	(d)	$800-500+100=$
(e)	$500+200-400=$	(f)	$700-100+200=$
(g)	$300+500-100+200=$	(h)	$600-200+100-300=$
(i)	$700-600+300-100=$	(j)	$900-400+200-700=$

Just the Answer (C)

Do each of these questions in your head and write down the final answer.

(a)	$1000+4000-2000=$	(b)	$3000+5000-1000=$
(c)	$6000+3000-5000=$	(d)	$8000-4000+2000=$
(e)	$5000+1000-2000=$	(f)	$7000-4000+1000=$
(g)	$3000+4000-2000+3000=$	(h)	$9000-6000+1000-2000=$
(i)	$7000-4000+3000-2000=$	(j)	$8000-4000+2000-3000=$

Just the Answer (D)

Do each of these questions in your head and write down the final answer.

(a) $10+50-30=$

(b) $400+500-200=$

(c) $6000+2000-1000=$

(d) $80-40+30=$

(e) $500+300-200=$

(f) $9000-5000+2000=$

(g) $20+40-30+10=$

(h) $7000-3000+1000-4000=$

(i) $900-400+100-200=$

(j) $50-20+30-40=$

Just the Answer (E)

Do each of these questions in your head and write down the final answer.

(a) $20+400+7=$

(b) $3+50+600=$

(c) $600+9+70=$

(d) $80+6+500=$

(e) $80+100+3=$

(f) $700+8+20=$

(g) $3000+400+2+50=$

(h) $90+6000+1+200=$

(i) $700+40+3000+5=$

(j) $8+4000+20+300=$

Just the Answer (F)

Do each of these questions in your head and write down the final answer.

(a) $9+4000+500=$

(b) $3000+5+40=$

(c) $60+3+5000=$

(d) $800+4000+2=$

(e) $500+1000+30=$

(f) $70+4000+100=$

(g) $3000+400+7=$

(h) $900+60+3000=$

(i) $70+4000+3=$

(j) $80+7000+200=$

Just the Answer (G)

Do each of these questions in your head and write down the final answer.

(a) $3000 + 4000 - 2000 =$ (b) $30 + 50 - 40 =$

(c) $60 + 300 + 5 =$ (d) $800 - 400 + 200 =$

(e) $50 + 1 + 200 =$ (f) $700 + 40 + 1000 =$

(g) $5000 + 4000 - 3000 + 1000 =$ (h) $90 - 60 + 20 - 30 =$

(i) $70 + 4000 + 3 + 200 =$ (j) $500 + 40 + 2000 =$

Just the Answer (H)

Do each of these questions in your head and write down the final answer.

(a) $200 + 500 + 100 + 3 =$ (b) $40 + 10 + 20 + 6 =$

(c) $200 + 500 + 30 + 10 =$ (d) $40 + 50 + 3 + 1 =$

(e) $300 + 400 + 50 + 10 =$ (f) $10 + 30 + 40 + 3 + 5 + 1 =$

(g) $200 + 200 + 40 + 30 + 10 =$ (h) $500 + 200 + 30 + 10 + 4 =$

(i) $500 + 300 + 30 + 4 + 2 =$ (j) $200 + 30 + 60 + 2 + 1 =$

Just the Answer (I)

Do each of these questions in your head and write down the final answer.

(a) $100 + 5 + 200 + 3 =$ (b) $50 + 6 + 40 + 1 =$

(c) $200 + 30 + 400 + 10 =$ (d) $30 + 5 + 20 + 2 =$

(e) $4 + 300 + 1 + 400 =$ (f) $20 + 700 + 40 + 100 =$

(g) $7 + 50 + 20 + 1 =$ (h) $4 + 500 + 300 + 2 =$

(i) $600 + 30 + 20 + 5 =$ (j) $200 + 200 + 30 + 7 =$

Just the Answer (J)

Do each of these questions in your head and write down the final answer.

(a) $1000+40+200+1=$ (b) $300+500-400+100=$

(c) $6000+30+2000+10=$ (d) $80-50+20-10=$

(e) $5+60+2+30=$ (f) $70+400+400+10=$

(g) $300+400-100+300=$ (h) $90+6000+5=$

(i) $70+500+20+200=$ (j) $80-60+20-30=$

Just the Answer (K)

Do each of these questions in your head and write down the final answer.

(a) $200+540=$ (b) $37+20=$

(c) $6025+3000=$ (d) $400+341=$

(e) $50+26=$ (f) $7040+1000=$

(g) $309+500=$ (h) $60+34=$

(i) $4000+2805=$ (j) $700+230=$

Just the Answer (L)

Do each of these questions in your head and write down the final answer.

(a) $20+740=$ (b) $3700+200=$

(c) $7024+30=$ (d) $40+541=$

(e) $50+210=$ (f) $8040+30=$

(g) $3190+500=$ (h) $60+520=$

(i) $400+2305=$ (j) $50+2310=$

Just the Answer (M)

Do each of these questions in your head and write down the final answer.

(a) $200+9300=$ (b) $370+600=$

(c) $4021+4000=$ (d) $40+632=$

(e) $50+26=$ (f) $540+50=$

(g) $1280+500=$ (h) $600+327=$

(i) $400+7201=$ (j) $231+600=$

Just the Answer (N)

Do each of these questions in your head and write down the final answer.

(a) $100+4+20+7000=$ (b) $40+50-30+10=$

(c) $600+70+200+20=$ (d) $400+267=$

(e) $30+537=$ (f) $7+400+500+1=$

(g) $700-400+100-200=$ (h) $6210+500=$

(i) $200+431=$ (j) $8000+50+1000+30=$

Just the Answer (O)

Do each of these questions in your head and write down the final answer.

(a) $5\times 3+2=$ (b) $4\times 7-2=$

(c) $9\times 2+1=$ (d) $1\times 4\times 3+1=$

(e) $5\times 2\times 2+3=$ (f) $7\times 2-4=$

(g) $3\times 1\times 3+2=$ (h) $5\times 6+4-1=$

(i) $8\times 3\times 0+4=$ (j) $4\times 4+2-3=$

Just the Answer (P)

Do each of these questions in your head and write down the final answer.

(a) $5 \times 5 + 3 =$

(b) $4 \times 6 - 3 =$

(c) $8 \times 2 - 1 =$

(d) $1 \times 2 \times 4 + 2 =$

(e) $4 \times 2 \times 2 - 2 =$

(f) $7 \times 3 + 4 =$

(g) $3 \times 0 \times 7 + 2 =$

(h) $5 \times 3 + 4 - 1 =$

(i) $4 \times 3 \times 1 + 2 =$

(j) $4 \times 8 + 2 - 1 =$

Just the Answer (Q)

Do each of these questions in your head and write down the final answer.

(a) $8 \div 2 + 3 =$

(b) $9 \div 3 + 4 =$

(c) $6 \div 2 - 1 =$

(d) $18 \div 2 - 4 =$

(e) $70 \div 7 + 2 =$

(f) $20 \div 4 \times 2 + 1 =$

(g) $32 \div 8 \times 3 - 2 =$

(h) $45 \div 5 \times 2 + 1 =$

(i) $36 \div 6 \times 5 + 2 =$

(j) $27 \div 9 \times 7 - 1 =$

Just the Answer (R)

Do each of these questions in your head and write down the final answer.

(a) $10 \div 2 + 2 =$

(b) $18 \div 3 + 1 =$

(c) $15 \div 3 - 1 =$

(d) $20 \div 2 - 3 =$

(e) $60 \div 10 + 2 =$

(f) $16 \div 4 \times 2 - 1 =$

(g) $36 \div 9 \times 3 + 2 =$

(h) $35 \div 5 \times 2 - 1 =$

(i) $30 \div 6 \times 5 + 2 =$

(j) $18 \div 9 \times 6 - 1 =$

Mental Computations

1. Below you will find eight different problems that were solved by eight different students. All eight students solved the problem in their head and got the correct answer. When asked how they solved their question so quickly, they provided the following mathematical statements to support their answers.

Camille's	Ryan's	Miya's	Renu's
$\mathbf{298+199}$ $300+200-3$ $500-3$ $\mathbf{497}$	$\mathbf{29+29+29}$ $(3\times 30)-3$ $90-3$ $\mathbf{87}$	$\mathbf{557-98}$ $(557-100)+2$ $457+2$ $\mathbf{459}$	$\mathbf{25\times 164}$ 50×82 100×41 $\mathbf{4100}$
Taro's $\mathbf{\frac{1}{2}\times 55\times 12\times \frac{1}{5}}$ $\frac{1}{2}\times 12\times \frac{1}{5}\times 55$ 6×11 $\mathbf{66}$	**Odell's** $\mathbf{\frac{7}{5}+\frac{3}{7}+\frac{4}{5}-\frac{2}{10}}$ $\frac{7}{5}+\frac{4}{5}-\frac{1}{5}+\frac{3}{7}$ $\frac{10}{5}+\frac{3}{7}$ $\mathbf{2\frac{3}{7}}$	**Shirley's** $\mathbf{4\times 38}$ $(4\times 40)-(4\times 2)$ $160-8$ $\mathbf{152}$	**Evan's** $\mathbf{5-1\frac{4}{11}}$ $\left(4+\frac{11}{11}\right)-\left(1+\frac{4}{11}\right)$ $(4-1)+\left(\frac{11}{11}-\frac{4}{11}\right)$ $\mathbf{3\frac{7}{11}}$

Using complete sentences, explain what each of the eight students did when solving their problem.

2. Solve these eight problems using the strategies you learned above. Although they can be solved mentally, I would like for you to provide the mathematical statements that you used to obtain your answers.

(a) $399+99+198$ (b) $24+24+24$ (c) $3421-198$ (d) 40×18

(e) $\frac{1}{3}\times\frac{1}{8}\times 27\times 32$ (f) $\frac{1}{2}+\frac{1}{4}+\frac{1}{9}+\frac{1}{4}$ (g) 3×198 (h) $7-2\frac{5}{6}$

3. Solve the following questions in your head. No work needs to be shown. Do not use a calculator or pencil and paper. Place the answer in the space provided.

(a) $697+199$ (b) $498+98+299$ (c) 5×59

Answer: _________ Answer: __________ Answer: __________

(d) $256 - 199$ (e) 4×97 (f) $49 + 49 + 49$

Answer: __________ Answer: __________ Answer: __________

(g) $52 + 163 + 48$ (h) $7\frac{1}{3} - 5\frac{1}{6}$ (i) $1363 - 298$

Answer: __________ Answer: __________ Answer: __________

(j) 79+79 (k) $1 + \frac{3}{4} + \frac{1}{2} - 0.25$ (l) $5.97 + 2.99 + 0.98$

Answer: __________ Answer: __________ Answer: __________

(m) $165 + 347 + 35$ (n) $5672 - 1999$ (o) $\frac{2}{5} \times 14 \times \frac{1}{7} \times 20$

Answer: __________ Answer: __________ Answer: __________

(p) $\frac{3}{4} - 0.5 + \frac{1}{4}$ (q) 18×5 (r) 38+38+38

Answer: __________ Answer: __________ Answer: __________

(s) 6×98 (t) $39 + 39 + 39 + 39$ (u) $4\frac{1}{2} + 7\frac{1}{4}$

Answer: __________ Answer: __________ Answer: __________

(v) $0.1 \times 18 \times 20 \times \frac{1}{3}$ (w) $5 - 1\frac{4}{9}$ (x) 27×30

Answer: __________ Answer: __________ Answer: __________

4. Not all questions can be solved in your head. Consider the various strategies that you encountered in questions 1 and 2 and then determine if each of the following questions can be solved using mental computation. If a particular question can be solved in your head, provide the final answer. Do not use a calculator or pencil and paper.

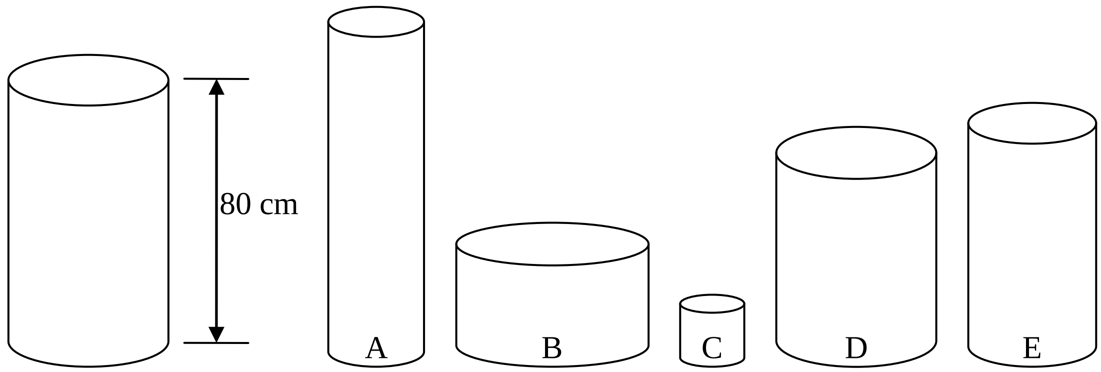
(a) $256 + 131 + 127$ (b) $40 + 39 + 37$ (c) 3×22.6

(d) $3 - \frac{7}{8}$ (e) $\frac{1}{4} \times 27 + \frac{1}{3} \times 10$ (f) $\frac{4}{5} + \frac{2}{3} - \frac{5}{7}$

(g) $142 - 68$ (h) $3.57 - 1.831$ (i) 4×14

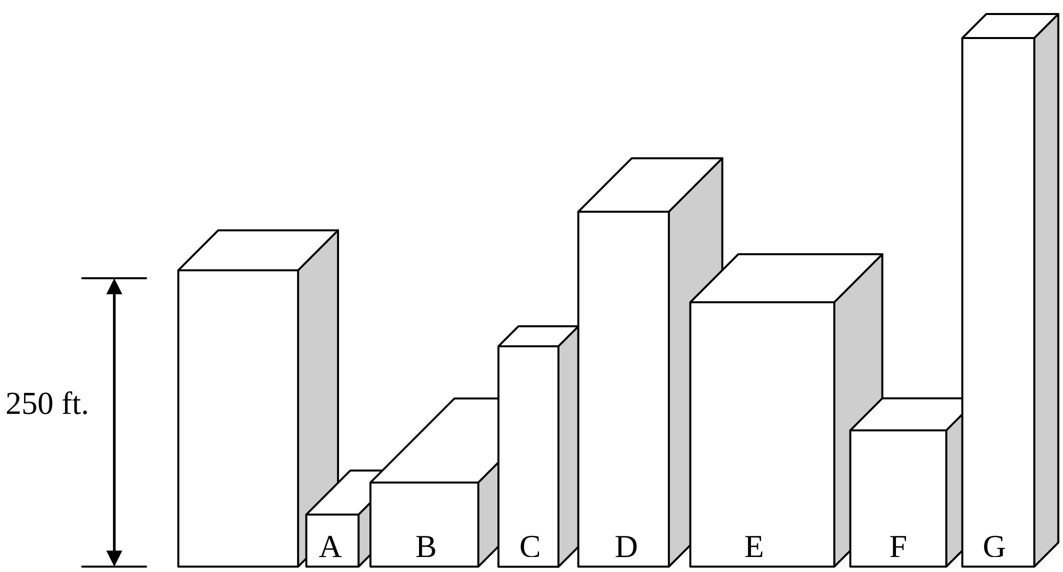
Estimating by Comparing Objects

1. Tanya has six cylindrical containers. She knows that one of the containers is 80 cm tall. Estimate the height of the other five containers based on the following scale diagram. Do not use a ruler.



Estimated Height of Container A: _______ Estimated Height of Container B: _______
Estimated Height of Container C: _______ Estimated Height of Container D: _______
Estimated Height of Container E: _______

2. Jacob is looking down a city block and comparing the heights of different buildings. He knows that the first building is 250 feet tall. Use the scale diagram below to estimate the height of the other buildings. Do not use a ruler.



Estimated Height of Building A: _________ Estimated Height of Building B: _________
Estimated Height of Building C: _________ Estimated Height of Building D: _________
Estimated Height of Building E: _________ Estimated Height of Building F: _________
Estimated Height of Building G: _________

Classify

1. There are several numbers and mathematical operations listed below. You must determine whether the number or statement is best classified as closest to zero, closest to one-half or closest to one. Fill the number or statement in the appropriate column in the chart. You are not permitted to use a calculator.

Closest to 0	Closest to $\frac{1}{2}$	Closest to 1

Numbers or Mathematical Statements

0.4507	$\frac{19}{37}$	49% of 2.2	$\frac{1}{5}\times\frac{1}{2}$	0.98^2
$\frac{2}{61}$	0.932	2 - 1.089	15% of 1	$0.72-\frac{1}{5}$
$\frac{1}{6}\times 0.4$	$\frac{10.2+11.4}{20.9}$	3% of 0.4	$\frac{15}{13}$	0.049+0.001
$\frac{1}{10}+\frac{3}{8}$	0.0997	$\frac{1}{15}$	97% of 0.95	$7.2\times\frac{1}{120}$
$\left(\frac{1}{3}\right)^2$	98% of 0.52	$\frac{2+1}{2+5}$	$12.2\times\frac{1}{13}$	10% of 4.8
$\frac{8}{7}-\frac{9}{8}$	0.01099	$\frac{5}{8}\div\frac{1}{2}$	$\left(\frac{10}{9}\right)^2$	$\frac{3}{4}\div 10$

2. For each of the three categories in question 1, create two numbers and/or mathematical statements that would be appropriate.

Reasonable Estimates?

Mr. Shah gave his students a series of estimation problems and he asked them to supply the first step they would use to tackle the problem. Their responses have been supplied below each of their names. You must decide which, if any, of these techniques would ultimately provide a reasonable estimation for the particular problem. Place a check mark or the letter x to the right of each name indicating your approval or disapproval with their technique. You are not permitted to use a calculator.

1. Estimate: $(3.1)^3 + \frac{8}{17} \times 49$

Manish		Danielle		Jamaar		Shiori	
$3^3 + 0.5 \times 50$		$9 + \frac{1}{2} \times 48$		$3^3 + 24$		$9 + 25$	

2. Estimate: $7.9 \div 0.24 + 10.07$

Manish		Danielle		Jamaar		Shiori	
$2 + 10$		$8 \div \frac{1}{4} + 10$		$32 + 10$		$8 \div 10$	

3. Estimate: $2^3 + \frac{25}{12} \times 9.6$

Manish		Danielle		Jamaar		Shiori	
$6 + 2 \times 10$		$2^3 + \frac{1}{2} \times 10$		$4^3 + 10$		$2^3 + 2 \times 10$	

4. Estimate: $0.41 \times 139 \div 0.52$

Manish		Danielle		Jamaar		Shiori	
$\frac{2}{5} \times 140 \times 2$		$\frac{4}{10} \times 140 \div \frac{1}{2}$		$\frac{1}{2} \times 150 \div 2$		$60 \div \frac{1}{2}$	

5. Estimate: $\dfrac{3.82 \times 6.1}{2.09 + 9.84}$

Manish		Danielle		Jamaar		Shiori	
$\frac{4}{2} \times \frac{6}{10}$		$\frac{4}{2} + \frac{6}{10}$		$\frac{4}{12} \times \frac{6}{12}$		$\frac{4 \times 6}{12}$	

6. Estimate: $\dfrac{4.89 + 8.06}{2.07 \times 2.95}$

Manish		Danielle		Jamaar		Shiori	
$\frac{5}{2} \times \frac{8}{3}$		$\frac{5+8}{6}$		$\frac{5}{6} + \frac{8}{6}$		$\frac{5}{2} + \frac{8}{3}$	

Do It In Your Head (Part 1)

Solve each of the following in your head. Calculators are not permitted.

Evaluate 32 + 99.	What is 100% of 6?	Estimate 19×31.	What is 2000 divided by 50?
Round 347 to the tens.	Find the number halfway between 11 and 15.	Write 6 and 42 hundredths in decimal form	Evaluate 18 + 25 + 2.
Double 3, add 5, and subtract 2.	Evaluate 70 + 70 +60.	3 dimes and 4 nickels is worth how much?	What is the largest number you can make with the digits 5, 6, and 2?
Evaluate $2 \times 7 \times 5$	Estimate 4.13 + 5.91	Which is the greatest? 2 + 3 2×3 2^3 3^2	If a dozen apples cost $1.40, how much would 6 apples cost?
How many $\frac{1}{2}$'s are in 3?	What is 50% of 18?	How many quarters are in $3?	Evaluate 5×32.
Evaluate 5.73 - 2.73.	Estimate 5.06×4.91.	Evaluate 37 + 198	Evaluate 31+ 30 + 29
Evaluate 59 + 59.	If 20 kg of potatoes cost $8, how much does 5 kg cost?	Which is greatest? 0.8 0.85 0.087 0.105	How many $\frac{1}{3}$'s are in 5?
Evaluate $4 \times 6 \times 5$.	What is 10% of 126?	Find the number halfway between 30 and 60.	Round 2.3461 to the hundredths.
How many dimes are in $4?	If $78 \times 5 = 390$, what is 79×5?	Evaluate $33 + 5 + 7 + 25$.	Estimate $12.07 + 13.9 + 8.16$.
Write 4 and 137 thousandths as a decimal.	Evaluate 450 - 299.	How many $\frac{1}{10}$'s are in 4?	Find the number halfway between 19 and 27.

Do It In Your Head (Part 2)

Solve each of the following in your head. Calculators are not permitted.

Evaluate 299 + 399.	What is $\frac{2}{3}$ of 12?	Evaluate 3×49.	Round 437.8 to the tens.
Double 20, add 35, and subtract 6.	Evaluate $\frac{1}{4} \times 30 \times 40$.	Which is greatest? $\frac{2}{7}, \frac{8}{9}, \frac{7}{6}, \frac{2}{3}$	Evaluate $\frac{2}{3} + \frac{3}{4} + \frac{1}{3}$.
If $243 \times 7 = 1701$, what is 244×7?	Estimate 16% of 40.	Evaluate $5 - 2\frac{3}{7}$.	Evaluate 498 + 299.
Evaluate 4×29.	What is 3^3?	Write 7 and 3 hundredths as a decimal.	What is 0.5% of 2000?
Evaluate $\frac{1}{5} + \frac{2}{3} + \frac{3}{5} + \frac{1}{5}$.	What is the number halfway between 71 and 101?	Evaluate 80+ 67 + 20.	What is $\frac{3}{5}$ of 35?
Evaluate 25×16.	Evaluate 80 + 80 + 73 + 80.	Triple $\frac{1}{3}$, add 4, and multiply by 10.	Evaluate 372 - 48.
Write 4 and 85 thousandths as a decimal.	Evaluate 5×32.	If $389 \times 4 = 1556$, what is 388×4?	Evaluate 19 + 19 + 19 + 19.
What is 35% of 80?	Evaluate $4 - 1\frac{7}{8}$.	Estimate 14% of 84.	What is 0.25% of 800?
Evaluate $3\frac{1}{6} + 4\frac{5}{6}$.	Which is greatest? $\frac{9}{10}, \frac{11}{12}, 0.88, \frac{3}{4}$	If $583 \times 3 = 1749$, what is 584×3?	What is $\frac{5}{6}$ of 30?
Evaluate $\frac{1}{4} + \frac{1}{2} + \frac{3}{5} + \frac{1}{4}$.	Double $\frac{3}{2}$, subtract 1 and multiply by -2.	Evaluate $\frac{1}{4} \times 6 \times 16 \times \frac{1}{3}$.	What is 15% of 140?

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